Introduction

This textbook was written for students of pedagogical faculty for all branches of study. It is divided into two parts: its first part gives a short overview of the history of mathematics from antiquity up to the most recent developments. It is obvious that the scope of this textbook does not allow for a detailed overview of the history, and therefore the author focused on those parts of history of mathematics that may be used at elementary and secondary schools. Apart from the facts themselves, examples of possible examples that may be used to present the historical material in schools are given. Second part contains short biographies of significant mathematicians, where preference was again given to those with whose names pupils are likely to meet during their mathematics lessons.

Part I

Selected parts from history of mathematics

Chapter 1

Mathematics of the antiquity

While historians who are mapping the development of mathematics in, say, the second half of 19th century have a relatively easy task, because it was then common for researchers to publish their results and these publications have been preserved in libraries, historians researching the development of mathematics face a very difficult task, as practically no sources have been preserved, had there ever been a sufficient amount of them. In any case, there is evidence that even people in Paleolite had mathematical knowledge. During archaeological excavations in Dolní Věstonice, the world-famous Moravian researcher, Professor Karel Absolon, found among other things also a wolf's rib bone with 55 notches, with each fifth notch longer than the others. Later, similar bones were discovered at other places. Bones that are in this way changed (bones with notches) are nowadays called *tally sticks*. Although there are still discussions going on as to what these bones were used for, it is very probable that they served for counting objects. After all, most contemporary waiters behave in a similar manner.

For contemporary youth, the books of adventures from Palaeolithic period written by Eduard Štorch, which were very popular among boys when I was young, might now be not as popular, but I will nevertheless remind the reader of one of them. The hero of the book Mamoth Hunters is a teenager called Kopčem, very thoughtful and very educated for his time, because he could count up to five, which nobody else, apart from the chief and a handful of top hunters, could do. When the number of objects was greater than five, it was said that there were many. Thus, in some sense, our $\aleph_0$ was known already to mamoth hunters, only their countable sets were all sets with the number of elements higher than five. It is interesting to notice that some natural nations, untouched by Euro-Atlantic civilisation, count in the same way until today. A similar example may be found in current natural languages: in Czech, we count jedna, dvě, tři, čtyři, etc. However, when we speak of a certain part of the whole, we have polovina, třetina, čtvrtina, etc. A similar curiosity may be found in other languages as well, for example the English count one, two, three, four and divide into halves.

While the Palaeolithic and Mesolithic man was a nomad, hunter, and picker, people in Neolith started to settle at one place, at least for a certain period of time, since their way of obtaining food had changed. From nomads and hunters, people became a settled farmer, people started building permanent residences, and human society began to break up into atoms, as people began to concentrate on a certain mode of working. First two great divisions of labour take place: cultivation of land was separated from shepherding and agriculture from the crafts. Construction of homes, production of tools and ceramics, all that suggests that mathematical, and especially geometrical, knowledge of people was being developed. As can be shown in archaeological findings, people decorated ceramics with geometrical patterns, and historians even denote the individual cultures through its ornaments, because we cannot know what these peoples were called (people with spiral ceramics, voluta ceramics, culture of bell-like goblets, etc.). With the accumulation of material wealth, the society could afford something that had before been unthinkable. Some people were excluded from the production process and devoted their time to spiritual activities. This, however, brings us to the period that is traditionally called antiquity and which is characterised by a change in the organization of the society, we speak of civilizations.

1.1 Mathematics of Ancient Egypt

Egyptian civilization originated at the lower reaches of the Nile, the origins of the Egyptian state being assumed to be around 3000 BC, when Upper and Lower Egypt were connected. The Egyptian empire founded by Meni can truly be called a thousand-year empire, because it lasted without interruption until 525 BC, when it was brought down by the Persians. In order to help us imagine how long this empire lasted, let us compare it with this country. If we deduct from today's date 2500 years, we reach the year 600 BC. Historians do not even know what the names of the tribes inhabiting this area were; Celtic tribes of Boja only came a century later. Renaissance of Egypt was taken care of by one of the military leaders of Alexander the Great, Ptolemaios, who founded the last royal dynasty of Ancient Egypt.

Ptolemaios empire, especially during the reign of the first Ptolemaians, prospered and became the centre of education of the contemporary world. In the newly founded capital of this country, Alexandria, Múseion (the House of the Muses) was founded by Démetrios from Faleras and in this institution, almost all of the greatest scientists of that time were working. Even the best known mathematical treatise of that time, Euclid's Elements, were written in this institution, generously funded by the king-farao Ptolemaios I. The name of this institution is preserved until today in the word museum, but the activity of this institution rather resembled the activities of our contemporary Academy of the Sciences. Over a short period of time, they also succeeded in building a truly large library in Alexandria, comprising hundreds of thousands of scrolls (rolls of parchment). It is said that if somebody came to Alexandria and carried a book them, this book was taken from them and deposited in the library, while the original owner had to be content with a copy only.

After the death of the last Queen of Egypt, Cleopatra VII,[1] Egypt became part of the Roman Empire. In the 7th century AD Egypt was conquered by the Arabs and they managed to finish what had been begun already by Caesar and in which the Christian Roman emperors successfully continued, led by the pious Theodosios, namely to destroy the remnants of the once famous Alexandrian library. The tradition has it that when Arab soldiers asked their military leaders what to do with the books, the answer was, "Burn them. Either they say something that is in Koran, and then they are useless, or they say something else, and then they are harmful."[2] In order to provide a balanced picture, we must say that such attitude was rather exceptional and held only for the early period of Islam. In the areas they conquered, science was well tended, however, the libraries founded by them were later a welcome object for vandalism, this time conducted by Christians.

For modern Europe, Egypt was discovered by a Napoleonic expedition towards the end of 18th century, which was not successful for France in the military sense, but was very important from the point of view of getting to know the culture of Ancient Egypt. Bonaparte namely brought with him also respectful French scientists, from mathematicians, Monge and Fourier took part in the expedition, among others. Although the French wanted to ship the findings back home in order to protect them from the uneducated fellahs,[3], the Europe of that time got to know about Ancient Egypt thanks to them and a new branch of science originated - Egyptology. By the way, Napoleon Bonaparte was one of the few modern monarchs who realised the importance of science for the society. Tradition has it that before the battle, he always commanded scientists and donkeys forward. When we realise how important donkeys were for the army, especially for transferring load, this command is not pejorative for the scientists, quite the contrary. This artillery officer and later emperor was interested in the most recent scientific discoveries and when prizes were awarded, scientist were not crouching at the back, but on the contrary, were sitting in the front rows. He himself was well versed in mathematics and could appreciate its importance.

Let us, however, return to Ancient Egypt. Although the Egyptians are said to be the most diligently writing people of the Ancient times and although they left us many written sources, unfortunately only very few mathematical texts have survived until the present time, perhaps because mathematical texts were not carved out in hieroglyphs into stone, but were written in hieratic script on papyri rolls. The preserved monuments, especially the pyramids, tombs, and church complexes, prove that mathematical knowledge of the Egyptians was wide-reaching, even though probably rather empirical in character.

The best known Ancient Egyptian mathematical text is the so-called Rhind Papyrus, which was written in 16th century BC, but whose draft is roughly three centuries older. It was named after the English antique dealer Rhind who bought it in 1858. Two of its large fragments are now in British Museum in London,

[1]This lover of the Romans Ceasar and Marcus Antonius is the main character of several novels, dramas and films and is apparently the best known Egyptian woman ever.

[2]The same story is passed around about the conqueror of Central Asia.

[3]In the end, they had to be content only with plaster casts and drawings ot the outstanding artist Denon; the originals were taken by the victorious English.

several smaller fractions have found their way to the Brooklyn Museum. It is composed of 14 sheets 38-40 centimetres wide and the length of the two bigger fragments added together is 513 centimetres. Moscow Papyrus was bought by the Russian Golenishchev at the end of 19th century and we will nowadays find it in Pushkin Museum of Fine Arts in Moscow. Its original length and width are 544 and 8 centimetres, but today, it comprises of one larger piece of 11 sheets and nine small fractions. Several fractions of a scroll with mathematical text were found in Káhún in 1889, two small fractions are owned by The Egyptian Museum in Berlin. In Egyptian Museum in Cairo, they take care of two wooden tablets with mathematical texts and by those, we have exhausted the list of Ancient Egyptian mathematical sources.

Let us now look what mathematical knowledge of the Ancient Egyptians has been preserved until today. The basic element of mathematics is a number and Ancient Egyptians worked with whole numbers and with fractions. For whole numbers, they developed a decimal non-positional notation, using digits from 1 to 1,000,000. Numbers were denoted through combinations of the necessary amount of digits for units, tens, hundreds, etc. In Egyptian way, the number 42 would be written in the following way: as one, one, ten, ten, ten, ten. Egyptians only used *unit fractions*, i.e. the reciprocals (multiplicative inverses) of natural numbers. Counting with fractions was thus rather tedious, because the result had to be a unit fraction again. In order to make the job easier, they designed tables in which fractions were expressed as a sum of several unit fractions, e.g. $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.

Ancient Egyptians knew basic arithmetic operations, multiplication and division being based on addition. One factor (multiplicand) was doubled as many times as it took to reach the second factor (multiplicand) and the relevant results were then added up. So should an Ancient Egyptian multiply 51 by 11 and did not have a calculator at hand, they would proceed in this way: 1x51 = 51, 2x51 = 102, 4x51 = 204, 8x51 = 408. If we add up the items 1, 2, and 8, we receive the correct result 561. On the other hand, in order to divide one of these numbers by the other, we double the divisor until we receive the dividend. 51 = 44 + 4 + 2 + 1, therefore 51:11 = 4 + 4/11 + 2/11 + 1/11. Raising a number to a power and extracting a root of a number is not very common in Ancient Egyptian texts, it is mostly performed on "decent" numbers, and it is probably that geometrical interpretations were used to perform these operations. Great care was devoted to calculation with unit fractions, but since detailed description of these operations exceeds the extent of this textbook, we refer the reader to other books for the treatment of this matter, for example [Vy]. As Ancient Egyptians solved solely practical problems, the computers had to be able to translate between the different units of measure.

We may deduce from the preserved mathematical texts that Ancient Egyptians had comparatively extensive knowledge of geometry, especially when it comes to its practical uses, i.e. mainly the calculations of areas and volumes. Ancient Egyptians could calculate the area of a rectangle. They only knew isosceles triangles and their areas were computed as the area of the rectangle with an equal area. The area of a trapezium was calculated by transforming the task to calculating the area of an isosceles triangle whose base was as long as the sum of the lengths of the two bases of the trapezium, in other words, they knew the formula $S = \frac{(a+c).v}{2}$. A circle was determined by its radius and its area was calculated by transforming it to a square of a roughly equivalent area. The length of the side of the square emerging from this transformation and the diameter of the circle were in the ratio $\frac{8}{9}$. Ancient Egyptians did not know the number π, but their results correspond to $\pi \doteq 3.16$.

As the pyramids, the only of the seven wonders of the Ancient World that has survived until these days and which probably also our descendants will be able to admire, Ancient Egyptians also had ver good knowledge of stereometry. They could calculate the volumes of a rectangular cuboid and a roller, in which they proceeded as we do nowadays, namely the area of the base is multiplied by the height. They could also calculate the volume of a truncated pyramid, but it is interesting that they had no special word for this solid and they only depicted it through in an ideogram. We could describe their procedure through he formula $V = \frac{1}{3}v(a^2 + ab + b^2)$. In Rhind papyrus, we can even find problems leading to the calculations of the slant angle of the pyramid. We will not find the substantiation of the procedure in the stereometry problems either and in contrast to the planimetry problems, not even a unified terminology was preserved. As the Ancient Egyptian mathematical documents originated relatively late, it is probable that their writers made use of the thousands of years of experience of Ancient Egyptian engineers.

In the preserved mathematical texts, tasks leading to arithmetic and geometric sequences may be found, among others tasks concerned with comparing the quality of beer and bread. Some tasks lead to solving equations and in some, the solution is found thanks to the method of false position. A small sample of Ancient Egyptian tasks is in the text below.

A triangle whose height is ten and base four. Tell me its area. Calculate a half of four, that is two, to determine the rectangle. Calculate with ten twice, the outcome is twenty. That is its area (Moscow papyrus, problem 4)

Method of calculating the area of a circle with (the diameter) 9 chet. What is its area? Subtract one ninth of it, that is one, and the remainder is eight. Calculate with eight eight times, you obtain sixty-four. That is its area on the surface. 64 secat. (Rhind papyrus, problem 50)

Method of calculating a sack with many precious metals. When it is said: a sack in which there is gold, silver, and pewter. This sack can be obtained for 84 shatei. What it is that corresponds to each metal, if a deben of gold gives 12 shatei, (for) silver it is 6 shatei and (for) pewter it is 3 shatei. Add up all that is given for a shatei of all metals, you obtain 21. Calculate with those 21 until you get 84 shatei. That is what you can get the sack for. You obtain 4. That is what you give for each metal. Procedure: calculate with four twelve times, you obtain: gold is 48, that is what belongs to it. 6x silver 24, 3x pewter 12. 21 altogether 84. (Rhind papyrus, problem 62)

Method of calculating 16 measures of Upper Egypt barley. Convert 100 loaves (of quality) 20, the rest for beer (quality) 2, 4, 6 $\frac{1}{2}\frac{1}{4}$ malt for dates. Calculate the share of the bread (of quality) 20, you obtain 5. Calculate the remainder from 16 for 5, you obtain 11. Do the division by 1 for these qualities obtaining $\frac{2}{3}\frac{1}{4}$. Calculate with $\frac{2}{3}\frac{1}{4}$ twice, because it was said $\frac{1}{2}\frac{1}{6}$ malt for dates, you obtain 1 $\frac{2}{3}\frac{1}{6}$..

Calculate with these 1 $\frac{2}{3}\frac{1}{6}$. until you find 11, obtaining 12x. Tell him: this is the corresponding beer. You found out correctly. (Moscow papyrus, problem 13)

Let us add a short commentary to the last problem. It is obvious that for 100 loaves of bread of quality twenty, we will use 5 heqats of barley. For a jug of beer of any quality, we will use $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$ of heqats of barley, which, after adding the dates, adds up to $\frac{11}{6}$ heqats. By dividing 11 heqats by this value, we will get the number of beer jugs. A careful reader has surely noticed that there are not twelve jugs, but only six. It is thus quite possible that the author of this problem liked beer a lot and in this case, it could have been wishful thinking.

1.2 Babylonian mathematics

Another area where ancient civilizations developed was Mesopotamia that is the area between the rivers Euphrates and Tigris. In contrast to Egypt, this area was inhabited by several nations in a sequence (Sumerians, Akkadians, Babylonians, Assyrians); to make things simple, we will talk about Babylonian mathematics, although this denomination is not quite accurate. In this area, the so-called cuneiform[4] was used, it was written on clay tablets, which were fired (in kilns), so a relative plenitude of sources for these empires was preserved. It seems that Babylonian mathematics was more advanced than Ancient Egyptian mathematics, at least this is what the preserved sources show. One of the greatest contributions of Babylonian mathematics was the use of positional notation for numbers. Babylonians used number sixty as the base, perhaps because the number 60 has many divisors. The division of the circle into 360 degrees and of the hour into 60 minutes and the like have their origin in Babylonian mathematics. These divisions are so strongly anchored in human minds that not even the creators of SI units, which is otherwise strictly based on decimal number system and its units are, apart from the exceptions, multiplied by a thousand or divided into a thousand parts, dared to divide an hour into a thousand of mili-hours and they only applied this division in the smallest unit, namely the seconds.

A hexadecimal number system was used already by the oldest people who inhabited Mesopotamia, namely the Sumerians. While for us, number 1011 is expressed as $1.10^3 + 0.10^2 + 1.10^1 + 1.10^0$, sometimes without being fully aware of this, the Sumerians apprehended this number as $16.60^1 + 51.60^0$. The Sumerians, however, did not know the zero, not even as a sign for an empty space. The peoples who came to the area between Euphrates and Tigris used the followed their example. The computer thus had to realise from the context of the problem which number the notation corresponds with. It was only towards the end of the Babylonian era that numbers with a sign for the zero appeared at the empty space. The Babylonians were excellent computers; the tablet YBC 7289 from the era of the famous king Hammurabi (around 1700 BC) contains a calculation of $\sqrt{2}$ with precision to one millionth. To make the calculations easier, they used, like the Babylonians, various tables. For example, Plimpton tablet number 322 contains a

[4]Cuneiform is the script that has been used for the longest period of time: it was used for more than three thousand years

number of Pythagorean triples.

The Babylonians were also experts in solving individual equations and their systems. Solving linear equations was considered trivial, but they also dared to solve quadratic equations and some types of cubic equations. As they did not know notation, they had to describe the individual operations in words. In the same vein, they did not consider negative numbers to be numbers, and thus the equations had to be given in such a way that they only contained positive numbers and negative solutions were ignored. We will demonstrate this on solving the equation $x^2 + x = 0.75$. The left side of the equation was completed to the square, namely $(x + 0,5)^2 = 1$, thus $x + 0.5 = 1$, therefore a $x = 0,5$. The other root, $x = -1.5$, is negative, but they were not looking for it, although it could easily be found this way $(x + 0.5 = -1 \Rightarrow x = -1.5)$.

An interesting problem can be found in tablet VAT 8398. *From (1) bur I harvested (4) gur of grain. From one other bur I harvested (3) gur of grain. Grain exceeds grain by (8.20). My field added and it gives (30.0). What are my fields?* This problem is written in a rather complicated way, it contains various units, which the Babylonians could transfer into another and back (in contrast to us), and hexadecimal notation is used for the numbers. To understand the Babylonian procedure better, we will formulate the problem in another way. *A box of bottles of beer was brought to the bricklayers. Kvasars cost CZK 15 per bottle, Velens CZK 12 per bottle. We paid CZK 276 altogether (without deposit). How many of each there are?* A Babylonian would have assumed that there were equal numbers of bottles of each kind, and then we would pay $10.15 + 10.12 = 270$ (CZK), so it is clear that there must have been more Kvasars than Velens, and he would have realised that by changing a Velen for a Kvasar would increase the price of the purchase by 3 crowns. It thus suffices to divide the difference between the calculated price and the given price by 3 and we immediately know that there were 12 Kvasars and 8 Velens. Today we solve similar problems with the help of a system of two equations with two unknowns, but according to the author, the *method of false position* is the most natural way of solving such problems and it is a pity that it is not mentioned in current textbooks. It would namely be no problem to substitute the beer sorts with lemonades, which are also produced in Černá Hora brewery, or with another produce suitable for children, or, in order to increase the awareness of prices of various goods, transform the problems into similar ones with contemporary units and prices.

Babylonians knew the formula for calculating the square of a binomial, the difference of two squares, the sum of first n members of a geometrical progression (geometrical sequence) with ratio equal to two, and the sum of the first n squares of natural numbers, although they only had at their disposal the description of the procedure in words and expressing it in a formula would probably be more precise. The above-mentioned Plimpton tablet proves that they could construct Pythagorean triples and that they were also aware of the geometrical interpretation of these numbers. As far as knowledge of geometry is concerned, however, they were slightly behind the Egyptians, or at least the currently known finding suggest that they could determine the area of a triangle and a trapezoid and that they knew approximate procedures for calculating the volume of a roller, a cone, and of a polygonal frustum. Peoples living in Mesopotamia, however, constructed their buildings with fired bricks, which were not preserved until today, except for the foundations.

Chapter 2

Mathematics in Ancient Greece

The Greek tribes, in several waves, settled not only in the area of contemporary Greece, but also the islands in the Aegean Sea and Anatolia (Asia Minor). It is precisely in the towns of Asia minor where we can find the origins of Greek, and thus also European, science. It is only since that time that we may speak of mathematics, Greeks were the first ones who started doing mathematics in the sense we know it. While before, only specific tasks were solved, the Greeks began to construct mathematics as a branch of abstract science, they began to generalize their findings and prove them. The mathematics of Ancient Egypt and Babylonia is for us practically anonymous, although we know some names, but in the case of Ancient Greece, we know the names of men whom we may call mathematicians in the contemporary meaning of the word. And these names are known even to people who have nothing to do with mathematics. The names of Archimedes, Euclid, Pythagoras, Thales, and others are generally known today. Their lives and especially work will be mentioned in particular in the second part of this textbook, but the results of their mathematical work cannot be missing from any short summary of the development of mathematics.

2.1 Figurate numbers

We will probably never learn why in Ancient Greece the face of mathematics changed so significantly from concrete problem-solving to an abstract science. According to some researchers, *figurate numbers* were one of the things that could have contributed to this. A natural number may namely be visualised in multiple ways - and the ensuing configurations are called figurate numbers (triangle, rectangle, square, pentagonal, etc.). Maybe it was a former soldier or physical education teacher who embarked upon the idea to order pebbles into couples and found out that sometimes, the resulting shape is closed (even numbers), and sometimes not (odd numbers). We will obtain the same result if we form one set of couples from two such sets, while it is irrelevant how large the original shapes were, but solely on the fact whether the original shapes were closed or not. This way, the first mathematical theorem was born and proved at the same time. On the other hand, I am not sure whether this finding was important for the working people, be it an architect or a tradesman with olives.

Triangular figural number can be expressed by the formula $a = \frac{1}{2}n(n+1)$, which is also the formula for determining the sum of the first n members of an arithmetic series, in which the first member is equal to 1 and the difference as well (i.e. $a_1 = 1$ and $d = 1$). Square figural numbers are determined by the formula $a = n^2$. When we look at their practical construction, we can see that the first such number is determined by one pebble. In order to get the next number, we have to add two pebbles in the horizontal direction and one in the vertical one, i.e. three in total. When constructing the third of one of square numbers, we add three pebbles in the horizontal direction and two in the vertical one, i.e. five altogether. This procedure is the visualization of the formula

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2$$

or, expressed in words, the sum of n odd numbers is equal to n^2.

2.2 Commensurability and the discovery of irrational numbers

From Ancient Greek legends, we may deduce that the Hellenic people were fond of music, and perhaps it was music which inspired Pythagoras and his pupils to create their special philosophical theory. Pythagoreans namely noticed that the tones formed by two strings sound melodiously precisely in those cases when the ratio of their lengths can be expressed in small integers. If we press the second string exactly at the half, we will get the octave (2 : 1), by shortening the string to two thirds, we will get the fifth (3 : 2), the fourth is represented by the ratio (4 : 3), etc. Why, then, should we look for the essence of the world in the fire, the water, or the indefinite apeiron, when we have the number. The essence of being is the number (arithmos), only the number will enable us to describe quantitative relationships of objects and phenomena. And just like the music will be melodious when the lengths of the strings are in the ration of small integers, the world will also be beautiful if we are able to express it in a similar way.

According to their imagination, there was the central fire in the middle of the universe, around which the Earth, the Sun, the Moon, the individual planets, and the stars rotate. As the Greeks only knew of five planets, it was necessary to invent also the Anti-Earth, so that there were ten spheres on which the heavenly bodies circulated. For Pythagoreans, the ten was a symbol of perfection, and thus they could not stand it if there were only nine spheres. The spheres were naturally perfect spheres, their radii were in ratios of small integers, and the motion of the celestial bodies was uniform. Just like music originates through the motion of the string or through vibration of the air column, there must have been music originating from the movement of celestial bodies, according to the Pythagorean imagination, beautiful, perfect music, which could only be perceived by great minds, since this kind of music can only be perceived through the reason. The Greek word kosmeo means beautiful, therefore the name cosmos and the word cosmetics are derived from the same root.

One was not regarded as a number, but as a basic building element of arithmetic. Natural numbers were collections of a certain number of units; even numbers were female, odd numbers male. Rational numbers were represented as ratios of natural numbers. Every two numbers were commensurable, because there always was the smallest number dividing both the original numbers. This happy state, however, did not last long, because irrationality (incommensurability) was discovered and the Pythagorean model of the world collapsed. It was prohibited to reveal this discovery to the public, but one cannot guide such secret forever, and soon also the people outside the community learnt about this discovery. Hippas from Metapont, who was said to have revealed the discovery, was even killed according to some legend, according to others, he killed himself, and according to yet others, he was punished by the gods.

The proof that the root of 2 is irrational belongs today to school examples of the application of proof by contradiction. If $\sqrt{2}$ was a rational number, it would be possible to write it down as a fraction, namely $\sqrt{2} = \frac{p}{q}$, for $(p, q) = 1$ (i.e. the greatest common divisor of the two numbers a and b is 1). After squaring both sides of the equation and reduction, we get $p^2 = 2q^2$, which means that p^2 is even, therefore also p is even, and can therefore be written as $p = 2k$, and after substituting it into the original equation, we deduce that q is also even, which is in contradiction with the assumption that p and q are mutually prime. This proof is simple, but the Greeks have discovered the incommensurability of $\sqrt{2}$ differently; we present the two most common possibilities in the text below.

Let a square be given, whose side is a units long and the diagonal u units long. At least one of these numbers is odd, because otherwise we could double the unit. According to the Pythagoras theorem, $u^2 = a^2 + a^2$, hence u^2 is also even, and thus also a half of the diagonal can be measure with an integer number of units. Through another use of the Pythagoras theorem, we get $a^2 = \left(\frac{u}{2}\right)^2 + \left(\frac{u}{2}\right)^2$, which means that a^2 is also even, which is in contradiction with the assumption. This proof is essentially a geometric modification of the proof given above.

Pythagoreans liked a lot the regular pentagon, and ascribed magical power to it. If we are to construct a regular pentagon$ABCDE$, it suffices to construct an isosceles triangle ABC and the remaining points are found easily. As according to the Pythagoreans, every two line segments were commensurable, it may be assumed that they were looking for such a measure. It can however be easily shown that if such a line segment exists, then it is also the common measure of the pentagon $A_1B_1C_1D_1E_1$, whose vertices are formed by the intersections of the diagonals of the original pentagon. In the same way, we could form smaller and smaller pentagons of the same measure as the original pentagon. This, however, is not possible, and so the assumption about the commensurability of the side and the diagonal is a wrong one and we have to accept that the opposite is true, that these two line segments are not commensurable.

With irrational numbers, three famous problems that the Greeks have been trying to solve unsuccessfully are also connected with irrational numbers. The first of these problems is the *duplication of the cube*. The following legend is associated with the problem: an epidemic of the plaque outburst on the isle of Delos, and as it would not stop, the inhabitants of the isle sent messengers to the Delf, asking them for advice. The response was easy—it was necessary to duplicate the pedestal of the statue of the god Apollonius, which was a cube made of gold. There was also the condition that the new altar also has the shape of a cube and it must be said that the inhabitants of the isle of Delos could not solve this problem. The gods, however, probably appreciated their earnest attempts, because the plaque epidemic passed in the end.

Another of these problems is *squaring the circle*, in other words, constructing a square with the same area as the given circle. While squaring the rectangle is an easy task, as it suffices to to use the theorem Euclid used in his proof of the theorem of Pythagoras $v^2 = c_a \cdot c_b$ and construct a right-angle triangle with the hypotenuse of the length $c_a + c_b$ and the height of this triangle is the side of the square sought, a similar construction for a circle could not be found. In contemporary fiction and journalism, squaring the circle stands for a problem that can only be solved with great difficulties.

The third of these problems is the *trisection of the angle*, in other words the procedure of dividing any angle into three equal parts using the ruler and the compass. Bisection, i.e. dividing the angle into two equal parts, is so easy that elementary school pupils should master this construction, but nobody succeeded in dividing the angle into three equal parts. Sometimes, the problem of *rectification of the circl*e, i.e. finding the line segment of the same length as the length of a circle, and the *construction of regular polygons*, are added to the three problems. All these tasks had to be solved by the so-called Euclidean construction, i.e. with the ruler and the compass only, which is through a construction of a finite number of lines and circles. Today, we know that these problems cannot be solved, because the number π is transcendental, or in other words, it is an irrational number which is not the root of any algebraic equation.

2.3 The Elements

Nowadays, we do not know much about the life of Euclid, but his work *Stoichea*, in English *The Elements*, granted him immortality. It is said that it is the second most often printed book in the history of mankind, and as the first position is taken by the Bible, The Elements have no serious competitor in the category of scientific books. In this work, Euclid concentrated all the mathematical knowledge of Greece of his time. It consists of 13 books, whose list and summaries may be found e.g. in [Eu]. There is a certain paradox in that The Elements are the most often read mathematical book in ever, they may also be labelled as the least readable one. The Elements consist namely only of definitions, axioms, theorems, lemmas, while each theorem is proved here. We will not find any examples in the book, neither motivating the theorem, nor explanatory ones, no connecting commentary; these are due to later mathematicians and allow better understanding of the text by the reader.

As the parts that are devoted to geometry, including the famous five postulates of Euclid (these will be introduced in the chapter on the 19th century), are cited quite often, let us rather focus on the arithmetical part of The Elements, i.e. at Books VII, VIII, and IX. At the beginning of Book VII, Euclid gives twenty-two definitions and he begins by defining the number one and the (natural) number (D7/1 and D7/2). He thus does not consider one to be a number, but a basic building stone of a number. We can deduce this from the fact that Euclid represents numbers by line segments; one thus represents the basic (unit) line segment and e.g. number five is represented by five equal line segments adjacent to each other. This notion of a number causes the clumsiness of some of the proofs, the attempts to regard one as an ordinary number were not successful.[1] The readers who feel that they have already read about that in this book are not mistaken, since Euclid took this part from the Pythagoreans.

We will now allow ourselves a small detour and cite a few phrases from the textbook [Sm1], published some 150 years ago. *The number (numerus, die Zahl) is several identical objects. One such object coming in that number is called a unit. In the number 7 kreutzers, the unit is represented by the kreutzer, in the number ten horses, the unit is a horse, and in the number five hundred, the unit is a hundred.* This quotation shows that the Priest Šimerka wrote his textbook along Euclidean lines and additionally, the example facilitates our understanding of the thinking of Greek mathematicians.

Let us, however, return to The Elements: further definitions concern common notions like even and odd numbers, prime and composite numbers, the smallest common multiple and the greatest common divisor, ratio, and others. The last, i.e. twenty-second, definition then introduces the *perfect number*, which is a number that is equal to the sum of all its parts (read: divisors smaller than the number itself). The book then continues with thirty-nine theorems concerning the divisibility of natural numbers. Here, we will also find the famous Euclidean algorithm, although not in the form known to us today. Theorem 1/7: *Two different numbers are given and if the smaller one is deducted from the greater and if the number that remains is never a divisor of the preceding one, and if we go on deducting until only a unit remains, then the original two numbers are mutually prime.*

If, for example, $a_1 = 19$ and $a_2 = 7$, then according to Euclid, we need to follow this procedure: 19-7=12, 12-7=5, 7-5=2, 5-2=3, 3-2=1. It is obvious how to shorten this algorithm and obtain its current version and the proof is then obvious, because it suffices to use the fact that from the condition $b|a_1 \wedge b|a_2$ it follows that $b|(a_1 - a_2)$. We will continue this way until we reach the contradiction that $b|1$. It is interesting to note that although this theorem is generally valid, Euclid's proof is based on only three deductions. Other general statements are, however, proved in the same way.

Book VIII contains 27 theorems concerning continual proportions or, if you like, finite geometric progressions. Let us give theorem 22/8 as an example: *If three numbers are continually proportional and the first one is a square, then the*

[1] E. g. the Stoic philosopher Chrysippos (280-207 B.C.).

third one is a square as well. The proof of this theorem is easy, because it holds that $a_2 = ka_1$, $a_3 = ka_2 = kk^2a_1$.

Book IX is devoted to the theory of parity and prime numbers. In it, we will find the theorems that have already been mentioned, concerning parity of the sum or product of numbers. In Theorem 22/9, for example, the following is stated: *If we add a certain number of odd numbers and that certain number is an even number, then the sum is also an even number.* The proof is presented in the following form: we take four odd numbers, subtract one from each of them, obtaining four even numbers. Their sum is an even number, while adding four ones to it does not change its parity.

In theorem 20/9, the following is stated: *There are more prime numbers than any given number of primes.* A proof by contradiction is given; Euclid takes three primes, finds their smallest common multiple and adds one to it. A number thus originating is either a prime (but that would be a fourth one), or a composite number. However, it cannot be divisible by any of the previous one, so we have forgotten about one. The Greeks did not have (our set-theoretical) notion of actual infinity, they only understood it as potential infinity, namely in the sense that they can increase the given number. For that matter, in the geometrical part of The Elements, the straight line is not defined as infinite, but as a line that can be extended without limits beyond either of the two endpoints.

Theorem 35/9 is in fact the formula for the sum of the first n members of geometric sequence: *Let any collection of numbers be given, all continually proportional. We subtract the first number from the second one and from the last one. Then the surplus of of the second in ratio to the first one is the same as the surplus of the last one in ratio to the sum of all the preceding ones.* This rather clumsy wording says, in modern notation, that $\frac{a_2-a_1}{a_1} = \frac{a_{n+1}-a_1}{a_1+a_2+\cdots+a_n}$. Deducing the well-known formula for the sum of first n members of a geometric sequence is then very easy.

The last theorem of this book (36/9) runs as follows: *Let us have any amount of numbers beginning with one which are continually proportional with the quotient two and let us add these numbers until their sum is a prime. If we multiply this sum by the last number [added], the product will be a perfect number.* Here, Euclid states a sufficient condition for a number of the form $(2^n-1).2^{n-1}$ to be perfect. It can be proved that this is also a necessary condition. Numbers of the form 2^2-1 are called *Mersenne* numbers and nowadays, roughly 40 of them are known that are also prime numbers and searching for Mersenne primes is nowadays a big hit for amateur mathematicians.

To conclude this chapter, I dare to present a brief contemplation concerning Euclid and the contemporary world. Euclid's Elements are based on the opinions of two well-known philosophers of the antiquity. The first one of them is Plato [2] and his idealist philosophy, which differentiates between two worlds, the real one, in which we live, and the transcendental world of ideas. In our world, there are people, animals, plants, and things that we get to know through our senses. These originate, change over time, and finally perish. A trembling pinscher is a dog, just like a mastiff is. If we disconnect a couple of carriages from a train, what will

[2] Plato (427/8-347 BC), Greek idealist philosopher, who founded the school called Academia.

depart from the station will still be a train. Transcendental world, however, can only be recognized by reason, is perfect and the only real one, and it is governed the idea of the Good. In this world, only one dog and only one train exist. The idea of the dog and the one of the train is not born and does not die, it does not reflect anything else in itself and it does not change into anything else either. The idea of the dog is naturally contained in all dogs that run around the earth, just like the idea of the train is contained in all trains that have ever traversed, are traversing, or will traverse the railroads. The idea itself, however, is not in any way influenced by the concrete realisation.

This is the way things are also in The Elements. Definition 1 states: *A point is that which has no parts.* Definition 2 tells us that *A line is a length without a width.* However, even if we sharpen a pencil with the most precise pencil sharpener equipped with the most modern technology, even then the points drawn with this pencil will be speckles and the line drawn magnified appropriately will resemble a motorway. As Vlasta Burian would say, geometry has abstracted itself from the reality. We solve geometrical through reason and then we more or less exactly demonstrate them on paper. It is thus no coincidence that the following inscription was there above the entrance to Plato's academy: *Do not enter, if you do not know geometry.*

The second one is Aristotle from Stagira,[3] the founder of logic. According to him, it is necessary to devote ample attention to the definitions of notions, because people communicate through notions. The road of recognizing the principles leads from from the individual to the general and it is called induction, these principles then have to be justified through judgement or deduction. If it is shown that the new principle is a consequence of already confirmed principles, then we have performed the proof. Not every statement can be proven, and such statements are called axioms. The Elements are constructed logically in this sense and other mathematical theories are also based on this principle

2.4 Practical Mathematics

From the previous lines, it might seem that the Greek scientists only solved theoretical tasks in the quiet environment of their studies. The Greeks, however, were also adept sailors, which can be seen from their mythology (Argonauts, the Odyssey, etc.), and also from the fact that they managed to colonize major part of the Mediterranean. There were seven wonders of the old world, as Philo said, and of those, five were on the territory inhabited by the Greeks and at least three (Temple of Artemis in Ephesus, Mausoleum in Halicarnassus, and the Lighthouse on the Pharos Island) were especially wonders of architecture. These facts are already an evidence of the fact that they could use mathematical knowledge in practice.

If we ask any passer-by about Archimedes, they will probably think about the object partially immersed in a fluid and will conclude that he was a physicist. If

[3]Aristotle (384-322 BC), leading personality of the Lyceum, polyhistor with great sense for the methodology of scientific examination.

we take into account current division of the sciences, we can of course agree with this statement, but we should not forget that Archimedes was also an excellent engineer and that from the work that has been preserved until today it is clear that we can also classify him as a mathematician. In any case, we can say that Archimedes was the most significant scientist of the antiquity. Pupils meet his name already at elementary school, so we will deal with some of his discoveries in more detail.

While the calculation of the perimeter or area of regular polygons does not present any difficulties, the determination of the formulas for the perimeter of a circle or the area of a circle is by far not easy and it requires much inventiveness on the part of the teacher to explain the formulas so that the pupils understand it, because the pupils have not yet acquired the necessary mathematical tools. It would nevertheless be worthwhile to return to this issue once after introducing goniometric functions and attempt to derive the formulas in a manner similar to the one used by Archimedes many centuries ago.

The method of Archimedes is based on the *exhaustive principle of Eudoxos*, which can be formulated in the following way: Let l be the length of a circle, l_n is the length of a regular polygon inscribed into this circle. Then for each $k > 0$ there exists n such that $l - l_n < k$. In a similar way, we can formulate this principle also for the length of a circle and the regular n-gon circumscribed to the circle and in a similar way, we can proceed also for the area of polygons inscribed or circumscribed to a disk.

Let us thus have a circle with diameter d and let us ask by what number we should multiply the diameter to get the length. If we denote that number by π, then we will obtain $l = \pi d$. For the circumference of a regular polygon inscribed to the circle we can easily derive the formula

$$l_n = nd \sin \frac{360^\circ}{2n} \tag{2.1}$$

.

By comparing both formulas, we obtain lower bound for the number π, namely $\pi > n \sin \frac{360^\circ}{2n}$. In a similar manner, we can obtain the upper bound because the circumference of a regular polygon circumscribed to the circle is

$$L_n = nd \operatorname{tg} \frac{360^\circ}{2n} \tag{2.2}$$

and it therefore holds that $\pi < n \operatorname{tg} \frac{360^\circ}{2n}$. Through connecting these inequalities, we obtain the bounds for number π, given by the following inequality:

$$n \sin \frac{360^\circ}{2n} < \pi < n \operatorname{tg} \frac{360^\circ}{2n}. \tag{2.3}$$

From these considerations, it is obvious that we can never obtain the exact value of the number π this way. For practical calculations, however, it suffices to know only an estimated value of this constant. It therefore suffices to determine the accuracy k in advance and to calculate the values of both estimates until we get $L_n - l_n < k$. If we are not calculating these values with the help of a computer, but only with a calculator, it is sufficient to start with a certain n and double the number of sides of the polygon until the required inequality holds.

We can proceed in a similar way when determining the constant by which we need to multiply the square of the diameter (radius) of a disk to obtain its area. For practical reasons, we will seek a number by which we need to multiply the square of the radius and we will again denote this number by π. The following formula can easily be derived for the area of an n-sided polygon inscribed into a circle:

$$p_n = nr^2 \sin \frac{360^\circ}{2n} \cos \frac{360^\circ}{2n} = \frac{1}{2} nr^2 \sin \frac{360^\circ}{n} \tag{2.4}$$

.

The area of a circumscribed n-sided polygon is then given by the formula

$$P_n = nr^2 \operatorname{tg} \frac{360^\circ}{2n} \tag{2.5}$$

. Thus, we have the following bounds for number π:

$$n \sin \frac{360^\circ}{n} < \pi < n \operatorname{tg} \frac{360^\circ}{2n} \tag{2.6}$$

and we proceed as before to determine number π.

Archimedes started from a regular hexagon and he stopped his calculations when he reached the regular 96-gon. It is usually stated that the used the value of $\pi = \frac{22}{7}$, which corresponds absolutely with the value $\pi \doteq 3.14$, which is a value sufficiently exact for the practical purposes in technical calculations.

Archimedes also worked on the quadrature of the parabola, i.e. calculating the area of a rectangular triangle whose hypotenuse is substituted for by the parabola $y = x^2$. In one of the ways, he used his knowledge of mechanics in an ingenious way. To illustrate his method, we will use current symbolic language to facilitate the understanding. Let us thus have a lever which we identify with the x axis and whose centre of rotation will be the origin of the coordinate system. On the left hand side, we will place an isosceles triangle whose sides will be $y = \frac{1}{2}x$, $y = -\frac{1}{2}x$ and $x = -1$. On the right hand side we then put the parabola $y = x^2$, where $x \in [0; 1]$. If we take any x_0 from this interval, then also the transversal line of the triangle has this length and the momentum of the force will be $M = x_0 \cdot x_0 \varrho g$. To achieve balance, we need to take also the transversal of the triangle on the left-hand side and place its end to the point $x = 1$. The whole triangle on the left-hand side can thus be equilibrated in such a way that we concentrate all the transversals of the triangle on the right-hand side in the point $x = 1$. As the density and gravity compensate each other and the triangle may be substituted for by a mass point placed in its centroid, we obtain the equation

$$P(x) \cdot 1 = \frac{x^2}{2} \cdot \frac{2}{3} x \Rightarrow P(x) = \frac{1}{3} x^2.$$

This, however, is not the only way in which squaring of the parabola can be achieved. Archimedes also used the exhaustive method and filled out the area under the parabola with triangles, see [Zn]. Apart from squaring the parabola, he also derived the formula for calculating the volume and surface of a cylinder and of a sphere. He also dealt with the issue of a sphere inscribed into an equilateral cylinder and found out that the rations of the volumes and surfaces of these solids are 2:3. This curiosity can also be easily derived when dealing with the topic of volumes and areas of solids already in elementary schools. When Cicero was a quaestor in Sicily, he found Archimedes' grave and discovered that a sphere inscribed into a cylinder was carved on the tomb, thus probably revealing how much the genius treasured his discovery. Evil tongues say that Cicero's discovery is one of the most important contributions of the Romans to the development of mathematics.

To end this part, let me add a few philosophical considerations. While Euclid's approach can be described as static and purely intellectual, Archimedes' methods can be described as dynamic and engineering-like (practical). It would probably be rather courageous to maintain that Archimedes used infinitesimal calculus in his considerations, but we can indeed find the basic ideas of this branch of mathematics in his work. This Syracusan was able to introduce mathematics into practice, as we have shown briefly, and perhaps because of that he was greatly admired by the author of the book [Be]. Here we can agree with the author of that book, but we do not agree with his despect of those scientists who belonged to the first kind. We think that the Euclidean constructions have their place in the teaching of mathematics; although only few people will have to, in their lives, construct a triangle given by a side and the median and the altitude to that side. Precisely these constructions, as well as proofs and similar tasks, play an indispensable role in the development of logical thinking.

Chapter 3

Mathematics in the Middle Ages

In 476, the last emperor of the Western Roman Empire, Romulus, was dethroned. It is a paradox of history that although his name was the same as the name of the mythological founder of Rome, the historians gave him the appellation Augustulus (mini-emperor). This date is often taken to be the breaking point between the Antiquity and the Middle Ages, although the eastern half of the Roman Empire did not break down under the raids of the barbarians and lasted for another thousand years. On the debris of the Western Roman Empire, empires of various Germanic tribes rose and fell. The most significant and also the most stable state was founded by the Franks, whose empire became the foundation of two big current European states - France and Germany. The most significant and the best known ruler of the Frank empire was Charles the Great, who understood the importance of education and on whose court Alcuin, the learned monk, lived. Charles's empire did not last long after its founder had died, but current leading European powers, France and Germany, were founded on its basis.

3.1 Alcuin and the others

If you take a book dealing with recreational mathematics, you will most probably find the following puzzle in it: *A man needed to transport across the river a wolf, a goat, and cabbage and he could not find another boat then such which could transport only two of them. However, he had been told that he should transport them all unharmed. If you know, advise how he could transport them unharmed.*

This problem is already around twelve hundred years old and I think that it will entertain even today and those who do not know it have to think for a while before they find the solution. It was first stated in the book *Propositiones ad acuentos iuvenes*, which can be translated as "Problems to sharpen the young", whose author is thought to be Alcuin himself. Apart from this problem, we can also find three other problems of this kind; Alcuin is namely thought to be the inventor of this kind of problems by historians of mathematics. Let us state one more of his problems, which is not so well known: *There were three men, each having an unmarried sister, who needed to cross a river. Each man was desirous of his friend's sister. Coming to the river, they found only a small boat in which only two persons could cross at a time. How did they cross the river, so that none of the sisters were defiled by the men?* From the solution given by Alcuin himself, it is clear that a sister would be defiled if she found herself alone with another man without her brother present. The solution to this problem implies that the requirements on the morals of women in the times of Alcuin were different from those of today.

A couple of years ago, the author came across a problem that is analogous to the previous one and which should allegedly be used in Japan as a test of intelligence during job interviews. The time allowed for solving the problem is half an hour, and as the author still works in Brno, it is clear that he has not solved it. However, there is a solution to the problem: *A father with two sons, a mother with two daughters, and a policeman with a captive. They have a boat for two people, which may only be driven by an adult. The sons cannot be left alone in the presence of the mother and the daughters cannot be left alone in the presence of the father. The captive cannot be left alone with any member of the family, but interestingly, he will not run away while not accompanied by the policeman.* Frank youngsters, however, did not sharpen their minds only through problems about carrying things over water, but also through others. A well-known monologue of Vlasta Burian about confusion among relatives (already Jaroslav Mošna entertained the public by similar ones) is modelled on a problem from Alcuin's book, namely the following one: *Two men, a father and a son, marry two women, a widow and her daughter, while the father marries the daughter and the son the widow. Tell me, I ask, what the relationship will be between the two sons born into these marriages.*

Prevailing part of the collection is, however, devoted to mathematical tasks. In it, we may find problems leading to equations with one unknown, geometrical problems, problems involving conversion of units and also tasks involving sequences as well as problems leading to Diophantine equations and combinatorial problems. Most of these tasks are similar to those we can find in today's textbooks, only the way of posing the problems is much richer than today's strict tasks. When we realise that in the 8th century, there was no algebraic notation as we know it today at hand, we have to admit that solving a word problem by the pupils of that time was a valuable achievement. It is, however, also quite possible that if a time machine transferred students of those times into the 21st century, they would not understand our approach either and might wonder why we solve such easy tasks in such a complicated manner.

Alcuin was not the only author whose collection of problems survived until these days. Four medieval problems are ascribed to Venerable Bede, of which the fourth one is interesting because it speaks of negative numbers. The collection ascribed to the Greek poet Metrodorus probably originated in 5th century Byzantium, as the problems were published in the Greek Anthology, written by Metrodorus. It is, however, also possible that the problems themselves were written by a different author and Metrodorus only versified them. A collection of six problems by Abu Kamil, called *Book of Rare Things in the Art of Calculation*, survived from the 10th century. The solutions to these problems are always found through a system of Diophantine equations and their formulation is rather stereotypic: a person has 100 drachmas and their task is to buy several birds for various prices. For more information, the reader is advised to consult [Ma1]. We present here a selection of tasks from this book as an advertisement.

Today's pupils would solve the following problem using a linear equation with one unknown: *A boy greeted his father: Be greeted, father, he said. The father answered, be well, son. May you live as long as you have lived up to now and may you triplicate this double quantity of the years and add one of my years and you will live for a hundred years. Let answer he who knows how old the boy then was.* Apart from mathematics, we should also appreciate how nicely the father and the son behaved to each other.

Today's tasks on the conversion of units are usually monotonously formulated through a brief command to convert decimetres to centimetres and a few tasks with concrete numbers follow. Alcuin, however, formulates these tasks with grace and the task chosen reminds one rather of the tales by Fontaine or Krylov: *A snail was invited by a swallow to lunch a league away. However, it could not walk further than one inch per day. How many days did it take for the snail to walk to that lunch?* The conditional in this task is appropriate, because according to Alcuin, snail would walk to the lunch in 90,000 days.

Problems about common work are usually set in a manner accurately described by Stephen Leacock in Literary Lapses. Superman A would almost complete the whole task by himself, B is a normal worker and C is so clumsy that one wonders how the two could have him on their team. Not so for Metrodorus: *(Dear) manufacturer of fired bricks, I very much wish to finish my house. Today the sky is cloudless and I do not need many more fired bricks; together there are only three hundred that are missing. That many you have made yourself in one day and two hundred were made in a day by your son and as many as that and fifty more were supplied by your son-in-law. How many hours does it then last when the three of you set out to work together?* Apart from the poetic value, we will also appreciate the realism of the problem. The son and the son-in-law help their father significantly, while it is understandable that the master is a better manufacturer than the journeymen.

Relations among the various teaching subjects can be strengthened by problems in which figures from Greek mythology are mentioned. Let us state one of them to end this section: *Once Aphrodite asked Eros, who came to her distressed. "What makes you so sad, my child?" He then answered: "I was coming back from Helicon, with a load of apples, but Muses have robbed me of those apples and then ran away. Clio took a fifth of them, Euterpe a twelfth of all the apples; then Thalia, the noble one, took an eighth, Melpomene a twentieth part, Terpsichore a quarter, and Erato took a seventh part as her share. Polyhymnia stole thirty apples, Urania then a hundred and twenty, Calliope stole away with a heavy load of three hundred apples. And then I came home to you - look - almost empty-handed; the goddesses only left me with fifty-five apples.* As the Muses have been and still are an inspiration for artists as well as scientists, we will forgive them this lapsus.

3.2 Fibonacci, or problems not only of breeding

Somebody placed a pair of rabbits at a certain place surrounded by walls on all sides to learn how many rabbits would be born during a year, if by rabbits it is so that a pair of rabbits gives birth to a new pair of rabbits every month and if rabbits begin to give birth when they are two months old. As the first pair gives birth to new rabbits in the first month, multiply by two, and in this month you will have two pairs, from these pairs, one, namely the first one, will give birth also in the coming month, so in the second month, you will have three pairs. From the three pairs, two will give birth in the third month, so you will have two more pairs in the third month and the number of rabbits in this month will reach five...

This problem can be found in the book *Liber abaci*, written by *Leonardo Pisano*, also known as *Fibonacci.* The author describes the situation in the individual months, until he finally comes to the conclusion that there will be 377 pairs of rabbits in that place at the end of the year. The solution to the problem ends with a note that we can count rabbits in this way up to infinity. If we disregard the slightly idealized conditions, as faithfulness of partners and sound health on one side and the unbelievably low and surprisingly regular reproduction, we have just been shown an interesting sequence. This sequence is nowadays called Fibonacci sequence and can be defined by a recurrent formula $F_k = F_{k-1} + F_{k-2}$, $F_0 = 0$, $F_1 = 1$.[1] The numbers in this sequence are called *Fibonacci numbers* and apart from mathematics, we can find them a.o. in Dan Brown's book Da Vinci Code.

Nowadays, we can meet Fibonacci numbers in various parts of mathematics, and we will mention an interesting connection here. In statement 11 of his Book II, Euclid solved the following task: *Divide a given line segment in such a way that the area of the rectangle whose one side is the original line segment and one of the parts is the same as the area of a square whose side is the other part.*

If x denotes the side of the square, then it must hold that

$$a(a-x) = x^2,$$

which leads to the quadratic equation $x^2 + ax - a^2$ and its solutions are $x_{1,2} = \frac{a(-1\pm\sqrt{5})}{2}$. Only the positive root fulfils the conditions of the problem. Let us denote

$$\Phi = \frac{a}{x} = \frac{x}{a-x},$$

then after substituting for x, we obtain $\Phi = \frac{1+\sqrt{5}}{2} \doteq 1,618034$. The problem of dividing a line segment into such parts is nowadays called the golden ratio and the number Φ the golden number. This number is also equal to the $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$. Further interesting details about this number can be found in [Ja], including more references to the sources.

Fibonacci was the most significant central European mathematicians and the sequence named after him is by far not his only contribution to mathematics. Apart from the already mentioned Liber abaci, he is also the author of four other mathematical treatises, which we will introduce by the translated titles: Practice of

[1] The name was first used by the French mathematician E. Lucas (1842-1891).

Geometry, Blossom, Letter of the undersigned Leonardo to Master Theodorus, the Imperial Philosopher, and Book of Squares. Although his works are not always original and although he often takes over the mathematical knowledge of other mathematicians, whose writings he probably encountered during his journeys, his work deserves attention. We cannot deal with his writings in detail here, the interested reader may consult [Be2], therefore we will only briefly introduce here some of his results.

Fibonacci was one of the first mathematicians to use Hindi-Arabic numerals and the decimal position system. Part of his work is devoted to the writing of numbers, numerical operations including finding the square root and conversion of units. In Liber abaci, we will find also problems solved by equations, linear as well as quadratic and cubic ones. As our contemporary notation was not used in those times and negative numbers were not calculated with, quadratic equations have to be divided into several classes, so that there are no negative coefficients. With quadratic equations, we can basically reconstruct the solution and conclude that we basically can use our formulas, but this does not hold for cubic equations, although the solution found by Fibonacci is very precise.

Leonardo Pisano worked also with sequences; apart from the already mentioned one which is named after him, we can find problems leading to the sum of the first n members of the arithmetic sequence, problems leading to the sum of squares, and the ever-green of recreational mathematics, namely the Ancient Egyptian task about cats. Since many years have passed since this problem was first posed, in his version, the cats became old women. In his works, we can find also problems solved through Diophantine equation. Apart from the problems about birds found at Abu Kamil, we can also find here more complex tasks, resembling rather Aritmetica by Diophantus. We can present this problem as an example: *Find a square number which, increased by 5 and decreased by 5, gives a square number.*

When defining basic notions in geometry, he followed Euclid. He also devoted time to "measuring figures", which is a notion under which we must see especially the instructions for calculating the area. He provided general formula for calculating the perimeter and the area of a circle; in his version, $\pi = \frac{22}{7}$. It is interesting that he reproduces also Archimedes' method for calculating the value of the π through circumscribing and inscribing a regular 96-gon. Fibonacci also provides instructions for the calculation of the volume of solids, especially the cone and the sphere. He also pays attention to the calculations of the lengths of regular pentagon and decagon and the golden ration. He was able to prove that the medians of a triangle meet in a single point and that this point divides each of the medians in the ratio 2:1.[2]

To end this section, let us mention two curiosities. In the works of Fibonacci, there are signs of using mathematical notation and of attempts to solve problems in a general way. We will present the following problem as an example: *Five horses consume six weighing units of oats in nine days. In how many days will ten horses consume sixteen weighing units of oats?* Fibonacci says that a horses consume b weighing units of oats in c days, and d horses e weighing units of oats in f days. Then it holds that $abc = def$ and from that equation, we can determine

[2]This fact was known already to Archimedes, but his proof has not been preserved.

the demanded value. Some problems are about money of several people; these problems lead to a system of linear equations with one negative solution. Fibonacci states that such a problem is solvable only when the given person is in debt (in other words, the solution is a negative number).

3.3 The Arabs

We have already encountered the Arabs in the part devoted to Egyptian mathematics and the note on their contribution to science was very negative. However, we have suggested that it was an exceptional case and now the right moment comes for us to introduce the reader to the Arabic contribution to the treasure box of world mathematics. When Mohammed fled from Mecca to Medina in 622, not many people probably suspected that he would victoriously return in eight hundred years later and that his descendants would conquer large territory on which the new religion - Islam - would be spread.

Over a few centuries, the Arabs conquered a territory that reached from Spain in the west and over North Africa and the Middle East up to the river Indus in the east. This empire was rather heterogeneous and it gradually broke up into several independent states and the empire gradually declined and lost the positions it had conquered. Similarly to the situation in the Roman Empire, native peoples continued to live on the territory conquered by the Arabs, and therefore the mathematics that was developed there is sometimes called Islamic mathematics rather than Arabic mathematics. However, this term is not exact either, as other religions, especially Christianity and Judaism, were tolerated on the territory. To make things simple, we will stick to the name that has traditionally been used in this country, namely Arabic mathematics.

Just like the Ptolemaic dynasty in Egypt, the caliphs of Baghdad were also aware of the importance of science and education and they supported scientific research in all possible ways. During the reign of Harun al-Rashid (786-809), a large library was established, which was continually supplied with new books. His successor, al-Ma'mun, then founded something like Mouseion. He called this institution *House of Wisdom* and concentrated top thinkers of his time in it. Similar institutions also originated in other places. In a simplified form, we can say that Arabic mathematicians acquainted themselves with the results of their colleagues in other cultures (Greece, Babylonia, India, etc.) and that they further developed the knowledge they thus learnt. We should be grateful to the Arabic thinkers that they translated Euclid's Elements, Diophantus's Arithmetics, the works of Archimedes, Apollonius, and other Greek scientists. Some of their findings would probably have been lost without the translations. Ptolemy's astronomical work Syntaxis megalé, for example, is much better known under its Arabic title Almagest. The translations were accompanied with thoughtful commentaries, which made these works much more comprehensible. The Arabs also freed themselves from geometric ideas in arithmetic.

One of the parts of school mathematics is algebra. This title comes from the Arabic and was taken from the title of the most famous Arabic mathematician, al-Khwarizmi, whose title is *A short book about the calculations through al-jabr and al-muquabala*. Apart from practical problems, (commercial contracts, testaments), he also treats solving linear and quadratic equations with integer coefficients in this book. Al-Khwarizmi does not use symbolic notation, and thus he has to describe all operations in words. He does not include negative numbers in his considerations, and therefore he has to consider several types of equations with positive coefficients. In the same way, he does not include negative solutions. When solving the equations, he uses two types of operations: al.-jabr, which basically consists in adding the same member to both sides of the equation, and al-muquabala, which consists in adding up the members of the same degree. Quadratic equations, however, were not the top achievement of Arabic mathematics. Omar Khayyam wrote a book on the classification and solving of equations of the third and fourth degrees and al-Kashi could solve some equations of fourth degree. In Arabic mathematics, attention was paid also to trigonometry, a discipline in which they reached considerable achievements. They drew upon the work of Greek (Ptolemy) as well as Indian mathematicians. The idea to measure the angle with the aid of the chord originated in Greece, and in Indian mathematicians improved it through the use of half of the chord, namely the sine function. The Arabs took over this knowledge and developed them. They introduced new functions, a shadow (cotangents) and a reversed shadow (tangents) and they also used the functions now half-forgotten, secant and cosecants. In order to simplify the calculations, tables of trigonometric functions were calculated. It is interesting that they only used this knowledge only for solving problems in astronomy. They solved problems in a plane similarly as the Greeks, namely by dividing the general triangle into two right-angle triangle. They proved the sine theorem and formulated the cosine theorem, without attributing it any significance.

The Arabs left one more significant trace in mathematics, one that we meet all the time, namely Arabic numerals. Already al-Khwarizmi realised how important it is that we can describe any quantity using only a set of nine symbols. There are actual ten symbols, since al-Khwarizmi uses decimal positional system of notation and for the sake of unambiguity, it is necessary to have a symbol for the free space. In his version, it was a circle, i.e. our later symbol for null. Decimal positional system, however, is not an Arabic invention, but an Indian one. The Indians have been gradually adopting this system since the 7th century. Other nations gradually took it over from them. As in India, in the Arabic countries, this system was only gradually, but steadily, adopted. It reached Europe through Spain, at that time governed by the Arabs (the Moresques). The oldest manuscript using Arabic (we should rather call them Indian) numerals is from 976 and it was found in northern Spain close to Logro?o. Decimal positional system gradually substituted mainly the impractical way of writing down numerals in the Roman way. However, as far as units of length and weight, which are mainly not based on the decimal system, are concerned, the tradition has largely been preserved until today.[3] At the end of this section, let us add a small linguistic note. Latin name of al-Khwarizmi was mostly Algoritmus and this name became the name of the new arithmetic.

[3]Even Good Soldier Švejk maintained that there should not be ten, but twelve flags in Konopiště, since in when in dozens, things are always cheaper.

Arabic name for the empty space, null, was *as syfr*. The word zero as well as the word cipher (now used in the sense of digit in Czech) originated from this word. Null comes from the Latin word nullus (none) and it permeated into the common speech of mathematicians in the 15th century.

3.4 Mathematics in the Czech lands

As was already mentioned, the first evidence of mathematical thinking, a notched rib bone, was found in Dolní Věstonice in southern Moravia. Let us have a look at how our predecessors developed mathematics during the middle ages. This will be difficult, as hardly any purely mathematical works from this period, and especially from the early Middle Ages, have been preserved. We therefore have to rely on material or, alternatively, indirect evidence. The first state on our territory was Great Moravia. All four rulers of Great Moravia were Christians and this religion was gradually adopted on the whole territory. In the towns of Great Moravia, churches were built and, as archaeological findings show, these were built in a certain module, and therefore it is certain that the old Moravians had sound technical knowledge. Old Moravians also had lively commercial relations with neighbouring countries, and as the systems of measuring and weighing units were then not at all uniform, the merchants of that time had to be well versed in unit conversions. The most important Christian holiday is Easter, but this is a movable feast. Given the limited communication possibilities, this feast was announced by the priest in each church, so they at least had to know the algorithm for the calculation of the dates.

Towards the end of 9th century, practically with the fall of the Great Moravian Empire, the house of Premyslids was on the rise and they soon reigned the whole of Bohemia and thanks to Břetitslav, also the territory of Moravia. Mathematical knowledge of this period can be described in a similar way to that during Great Moravia. The development of feudalism coerced further developments, especially in geometry. It was namely necessary to determine exactly the area of fields and forests, and by the same token, it was necessary to divide exactly the land when founding villages and towns. Surveying thus became an important branch of human activity and its level was very good.[4]

We know from the legends about Saint Wenceslas that the future prince and saint Wenceslas attended the school at the church of St. Peter and Paul in Budeč. This school educated the sons of the foremost noblemen, especially in order to prepare them for the duty of the priest. Similar schools were founded also at other churches. The school at the chapter of St. Vitus, where *studium generale minor* was founded, was the most famous one. The knowledge graduates from these schools obtained were on a par with the standard in Central Europe at that time. They learned to add and subtract, halve and duplate, multiply and sometimes also divide. Teaching to calculate with fractions and algorithms was less central.

[4]The most significant proof of the level of Czech surveying is New Town in Prague, founded by Charles IV. in 1348. The king and his surveyors outran their time by several centuries. Cattle Market (Charles Square) became the largest square in Europe and it kept this primacy until these days.

The range of natural numbers was approximately one million. In geometry, basic notions like point, line, surface etc. were used. The procedures to calculate the perimeter and area of basic figures in the plane as well as for the calculation of the volume of some solids.

The founding of the university in Prague in 1348 was a significant impulse for the development of education (not only) in Bohemia. Charles IV thus succeeded in fulfilling what his grandfather, Wenceslas II, tried to accomplish in vain, namely to found universitas magistrorum at studentium. Prague, nowadays Charles, university was the first one of its kind north of the Alps and students from other central European countries, not just from the Lands of the Bohemian Crown, studied there. We will be primarily interested in the faculty of arts whose role was different from the faculties of arts today. Philosophical (artistic) faculty was a sort of a preparatory course for the study at the faculties of medicine, law, and theology, and among other things, mathematics was also taught there. This faculty may also be called the faculty of seven liberal arts: the humanities-oriented trivium (grammatics, rhetoric, and dialectics) and the sciences-oriented quadrivium (arithmetic, geometry, astronomy, and music).

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Since the foundation of the university, many teachers took part in the teaching, of whom we will mention two who were probably the most important. The first of them is Master Christianus de Prachaticz (?1368-1439), a contemporary and a friend of Jan Hus and also a vicar and a priest at St. Michael church in the Old Town in Prague. His work is mostly not original; he adapted the work of foreign scholars (Sacrobosco, Johannes vom Erfurt). *Algorismus prosaycus magistri Chrisitianni*, in which he describes basic arithmetical operations, is considered the most important. His *Computus chirometralis* is devoted to composing the calendar, determination of movable feasts, cycles of the Sun, etc. His other works deal with astronomy and also with medicine.

The name of another important mathematician is known even less, even though the inhabitants of Prague may daily see one of his creations and that this work is also admired by many tourists from the Czech Republic as well as from other countries. This work is Prague horologue, which consists not only of the tower clock, but also of a planetarium. According to Alois Jirásek and his Ancient Bohemian Legends, this outstanding piece is the work of the genius named Hanuš from Rose, who was blinded by Prague councillors so that no other town could boast with such a beautiful horologue. This legend is romantic, but it is far from reality. The horologue was constructed by the foremost Bohemian clockmaker and mechanic, *Nicolas from Kadaň*. He, however, lacked the necessary knowledge of astronomy, so it is clear that he had to construct the work according to an outstanding astronomer. This astronomer was nobody else than *Jan Šindel (1373-?1450)*. His astronomical works have not been preserved, but they must have been outstanding, because they were also valued by world-renowned astronomers of the Rudolphine times, Kepler and Brahe. Like Christiannus, he also devoted attention to medicine.

Palacky considers the Hussite period the most important one in Czech history. He definitely had good reasons for this statement, since apart from success in the field of politics and the military, the Lands of the Bohemian Crown were also known for their religious tolerance, which was a phenomenon unheard of during the middle ages, but this period was not very favourable for the development of mathematics and the sciences. It cannot be said that mathematics was not developed at all, but its level remained basically the same until the time when the eccentric emperor Rudolf II moved to Prague. We will describe the developments during that time in the next chapter.

3.5 More names

So far, we have only introduced names and works that can be used in teaching mathematics at elementary schools and high schools. However, the Middle Ages in Europe were not only a period of stagnation of science and apart from the thinkers already mentioned here, there were others whose contribution to mathematics was not insignificant. In the following brief account, we will introduce a few other names. *Anitius Manlius Boethius (around 480-524)*, also known as the last Roman, worked in the court of the king of the Germanic Ostrogoths, Theoderic the Great. He was the author of textbooks for all the disciplines of quadrivium and thanks to him some knowledge of the Greek mathematicians was preserved to these days. His work was namely not original, but consisted in translations of Ancient Greek authors (Euclid, Nicomachus, etc.). His translation of Arithmetics by Nicomachus was used as a textbook for nearly a thousand years.

Gerbert of Aurillac was one of the greatest thinkers at the turn of the second millennium. The date of his birth is not known; he was from a poor, but free family. He was brought up and educated by the Benedictines from a nearby monastery. His interests were very wide: during his travels, he for example met with the work of Arabic scholars. His mathematical work was not original, but it contributed to the spread of mathematical knowledge in Europe. He contributed to the renaissance of the abacus, which he had improved. He was probably the first European who got acquainted with Arabic mathematics and to a limited extent used Arabic numerals. [5] His work *Geometria* deals with solving some geometrical problems,

[5] On his abacus, we can find symbols similar to the Arabic gobar numerals. In his written work, however, he writes numbers out in words or he uses Roman numerals.

but the solution is given without proofs or detailed explanations. Some chapters in this work are also devoted to surveying. He was also an outstanding astronomer, in Lateran palace in Rome; he allegedly had an observatory constructed. He was also a major European politician and an advisor to Emperor Otto III, who aspired to unify the Christian world under one emperor and pope. In his church career, he reached the highest goal, in 999 he was the first Frenchman to be elected pope and he accepted the name *Silvester II.* After four years in the office, he died and the circumstances of his death are not clear. He is buried in the basilica of St. John in Lateran. There are many legends about his life, not always favourable for his role as a priest. As far as the author knows, mathematicians, in opposition to other professions, do not have a patron saint. Gerbert would probably be the best candidate for this position, but he was never canonized.

Nicole Oresme (?1323-1382) was the most significant personality of European mathematics in the 14th century. His most important mathematical work is *Algorismus proportionum.* In this work, he introduced the powers with positive rational exponent and stated rules for calculating with such expressions. In his *De latitudinibus formarum*, he applied dependent variable (latitudo - width) against the independent variable (longitudo - length), which can be changed. It is thus some kind of transition from coordinates on the heaven or earth sphere, which already Ancient scholars used, to geometrical coordinates as we know them now. It is probable that this work influenced also the founders of analytic geometry.

Johannes Mű̃ller called *Regiomontanus*Johannes Müller, lived in the 15th century. This excellent computer, but also a mechanic and a typographer also participated on translations and editions of classical mathematical manuscripts. His main work is called *De triangulis omnimodus libri quinque*, which is a systematic introduction into trigonometry. Regiomontanus could not use our contemporary symbolic expressions, and therefore he had to formulate all the statements in words. Thanks to his work, trigonometry became independent of astronomy. He also compiled tables for the sine function for intervals of one minute with the radius 60,000. He could not free himself from understanding the sine as a half of the length of the chord for double angles. Values understood in this way, however, depend on the length of the radius. [6]

Voskovec and Werich, in their song from the play Těžká barbora (Heavy Barbara), were rather uncomplimentary about the Middle Ages. Also in this period, however, science developed, although especially in the narrow circle of educated members of the church. Mathematics usually did not exceed the level set by the Ancient Greeks, but in contrast to them, people in the Middle Ages valued mathematics especially for its practical applications and its development was to a certain extent influenced by the development of production forces. For more detailed treatment, I recommend the readers to consult [Be2], where we can find more details about some important mathematicians, extracts from their works including commentaries, and matters of interest concerning the science and education in the Middle Ages.

[6] Unit radius was introduced by Euler only in 1748.

Chapter 4

Mathematics in the 16th and 17th Centuries

Historians have not agreed about when the Middle Ages end: sometimes, it is stated that it ends with the discovery of America, other times, with the invention of printing. Both these events were significant milestones in the history of mankind, especially the latter. Gutenberg's invention made it possible that education gradually ceased to be the privilege of a small number of people. The discovery of the new continent then inspired voyages across the ocean and the trade significantly increased. For travels across the ocean, bigger ships were necessary as well as more reliable navigation. The legendary voyage of Ferdinand Magellan proved that the Earth is round, Copernicus came with heliocentric theory, which was further improved by Kepler and Newton then showed that it has to be so, from the point of dynamics.

These and other circumstances also served as an impulse for the development of mathematics. It now not only could compete with the knowledge of the Ancient nations, but also surpassed it and its development proceeded forward in an unstoppable manner. Complex numbers were discovered, foundations of the infinitesimal calculus were established, the discovery of logarithms significantly simplified especially calculations in astronomy, probability calculus brought chance to mathematics, algebraic notation significantly simplified the writing of mathematical texts, and we could proceed further in this manner. We will deal in greater details with those areas that can be used in school mathematics.

4.1 Algebraic equations, complex numbers

In one of the first printed mathematical books, *Summa de Arithmetica*, written by the Franciscan *Luca Paciolli*, we may learn, among other things, that solving of cubic equations is, given the state of mathematical knowledge of that time, approximately as impossible as squaring the circle. Already Ancient Babylonians could solve some special kinds of cubic equations, partial success was reached by the Ancient Greeks or the Arabs, but the general formula, or better expressed, and the general procedure was not found. Said with the occultist cook Jurajda, the mathematicians in renaissance Italy were predestined to find this solution.

The first of them was *Scipione del Ferro*, who lived in Bologna from 1465 to 1526. He found the solution for some kinds of equations,[1] however, he did not publish his solution, only communicated it to a few of his friends. Venetian computer *Niccolo Fontana (1499?-1557)*, known under his nickname *Tartaglia*, in English the Stammerer, allegedly could solve all types of equations, but he did not publish his results either and only shared them, under the oath of keeping silent, with the Milan physician *Geronimo Cardano ((1501-1576)*. This man, apart from medicine, devoted his time also to mathematics and contrary to his colleagues named above, he decided not to keep silent about the solution and he published his results and those of his collegues in an outstanding book *Ars Magna*, in which he explained the methods of solving equations of the third and also of the fourth degree. Contrary to some thinkers north of the Alps, Cardano honestly stated which results were his own and which should be attributed to his colleagues. In this book, we can thus read that Tartaglia shared the method for finding roots of the equations of the third degree, but without a proof, which motivated him to find the proof and publish it. This procedure, which seems to us very honest, however, made Tartaglia very angry and a quarrel started between both men, in which neither of them stayed away from rude words. In any case, the formulas for solving equations of third degree are called Cardano formulas in contemporary books.

These formulas are hardly used nowadays, we will nevertheless take up some space to present them here. If the equation $x^3 + a_1x^2 + a_2x + a_3 = 0$ is given, we can eliminate the quadratic member through substituting for $x = y - \frac{a_1}{3}$. We can thus only consider equations in the form $x^3 + px + q = 0$, whose roots are given by the formula

$$x = \sqrt[3]{\sqrt{\frac{p^3}{27} + \frac{q^2}{4}} + \frac{q}{2}} - \sqrt[3]{\sqrt{\frac{p^3}{27} + \frac{q^2}{4}} - \frac{q}{2}}.$$

When solving the equation $x^3 - 6x + 4 = 0$, we will see that it does not have a solution in real numbers, because the expression under the square root is negative. However, without using any specialised methods, we can confirm that the number 2 solves the equation. Mathematicians solving these equations then could not explain this paradox and chose to call it *casus ireducibilis*. Such a case occurs whenever the equation has three different real roots. We can easily confirm that the given equation has also two irrational roots, $-1 \pm \sqrt{3}$. Cardano also considered negative roots, which he called fictitious. In his work, he also introduced methods for solving equations of the fourth degree, which was a discovery of his pupil *Lodovico Ferrari (1522-1565)*.

To all probability, casus ireducibilis instigated the origin of *complex numbers*, because it felt that even a square root of a negative number could have some mean-

[1] As negative numbers were not fully accepted then, and mathematicians wrote the equations in such a way that the coefficients be positive. They did not use contemporary notation either.

ing. Already Cardano suggested this when he published the following problem, which we will state using today's symbolic: *If we are to divide 10 into two parts whose product is 30 or 40, it is clear that this case is impossible. Let us, however, proceed in this way: if we divide 10 into halves, a half is five; five multiplied by itself is 25. If we then subtract from it the required sum, say 40, (-15) remains; if we take a square root of this and add it to 5 and subtract it from five, we will get two numbers which, multiplied by each other, will give 40; these numbers are* $5+\sqrt{5}$ *and* $5-\sqrt{5}$. It is not clear how to explain the appearance of this problem, and researchers are inclined to think that Cardano wanted to illustrate by this example that even when solving quadratic equations, we might need to calculate the square roots of negative numbers. If it is so, the thought was revolutionary, because for the whole three millennia when quadratic equations were solved nobody thought this way. As we can easily see, we could solve the problem as a quadratic equation $x(10-x)=40$. He called square roots of negative numbers *quantita sophistica*, but he did not get far in determining their properties. *Rafael Bombelli (1526-1572)* was a different kind of man, because he himself stated eight basic rules for calculating with a complex unit, or, more accurately, he stated how a unit and an imaginary unit should be multiplied and how to multiply imaginary unit with itself. It is obvious from the formulas he published in his *L'algebra parte maggiore dell Aritmetica* that he already understood complex numbers as we understand them today. As an example, we can give the equality $\sqrt[3]{52+47i}=4+i$.

Complex numbers were relatively slowly implemented in mathematics, as their interpretation was not clear. The well-known mathematician and philosopher Descartes was the first one to use *imaginary* as the name of the number, from which later denoting the complex unit as i arose, introduced by Euler. He was the one who started expressing complex numbers in goniometric form and is also the author of the well-known identity $e^{i\varphi}=\cos\varphi+i\sin\varphi$. That seems to suggest that he understood complex numbers as points in the plane, but he did not publish that explicitly. The first one to understand complex numbers definitely as points in the plane was the Norwegian cartographer and geodesist *Caspar Wessel (1745–1818)*, main share on spreading this idea is attributed to *Carl Friedrich Gauss (1777-1855)* who in his dissertation used complex numbers to prove the fundamental theorem of algebra and in this *Theoria residuorum biquadraticorum* gave geometrical interpretation of these numbers in the way we understand them today. It can thus be said that naming the complex plane after this famous scientist is justified. That, however, is not true for the equality $(\cos\alpha+i\sin\alpha)^n=\cos n\alpha+i\sin n\alpha$, which is nowadays known as Moivre theorem. The French Hugenot *Abraham de Moivre (1667-1754)*, who spent most of his life in England, did indeed work with complex numbers and used this inequality, but he never wrote it in this way. Let us remember also *Lazare Carnot (1753-1823)*, who christened these strange numbers with the nowadays used name complex.

From this review, it is apparent that complex numbers were successfully used in solving (not only) mathematical problems, although the theory around them had not been elaborated yet, As an example, we can mention solving linear differential equations with constant coefficients, for example the equations describing the behaviour of a simple electrical circuit with an inductor and a capacitor has the following form: $L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0$. Characteristic equation has complex roots, so the solution of this equation is therefore $I = C_1 \cos\frac{1}{LC}t + C_2 \sin\frac{1}{LC}t$. We can determine concrete values of these constants, when the initial conditions are given. It is obvious that the dependency of the current on time is given by harmonic function and that harmonic oscillation takes place. This procedure may of course be applied also to mechanical oscillation. Solving of differential equations is usually not taught at high schools, but we can also describe what happens in alternate current circuits, like determining the impedance and phase difference (phase offset) between the electrical current and voltage.

4.2 Infinitesimal calculus

This notion stands for two operations, and those are the derivative and the operation in some sense inverse of the derivative, the integral. In English, these operations are usually covered under the name of calculus. While some parts of mathematics originate only in the modern times, the integral in particular originated in the Antiquity. The integral namely allows us to determine the area of a geometrical form with curved lines as its boundaries; the problem is thus a rather practical one. As we have already said, already the Ancient mathematicians could compute the area and volume of some such formations. They did not use the concept of function, but the basic idea of the integral, namely the division of the solid into parts whose volume we can compute and then adding these up, can be found already in their work.

Let us have a look at how Pierre Fermat approached the problem of computing the area of a part of a parabola given by the function $y = x^2$ when $x \in [0, 1]$. He took any number $q \in [0, 1]$ and constructed the sequence of points $1, q, q^2, q^3$, etc. on the x axis. In these points, he constructed lines perpendicular to the x axis, intersecting the parabola at points $[q^i; q^{i^2}]$. He then started to circumscribe rectangles whose one side were two neighbouring points on the x axis and the other was the corresponding value on the y axis. When we add up the area of all these rectangles, we obtain

$$1 \cdot (1 - q) + q^2(q - q^2) + q^4(q^2 - q^3) + \ldots =$$

$$(1 - q) + q^3(1 - q) + q^6(1 - q) + \ldots = \frac{1 - q}{1 - q^3} = \frac{1}{1 + q + q^2}$$

The closer q comes to number 1, the closer the area of the relevant rectangle is to the area of the parabola and also to $\frac{1}{3}$. From the above-stated procedure, it is clear that Fermat use the approach of limiting sequence, although only intuitively, this notion will still wait to be discovered for another two centuries.

Bonaventura Cavalieri (1598-1647), one of the best disciples of *Galileo Galilei*, chose a different approach. Cavalieri used the method of *cross sections*, i.e. cutting the plane into segments of infinitely small width; in the concrete case of the parabola, these parts have the length x^2 and infinitely small width. Cavalieri sought for the sum of these squares in the triangle ACE, while he supposed that all the transversals with the side AC are exactly those squares. Then

$$\sum_{EA} XY^2 = \sum_{ED} XY^2 + \sum_{DA} XY^2,$$

where D is the midpoint of line segment AE. Let us choose any point X on the line segment DA. The perpendicular line from point X to the leg AE will intersect the midpoint transversal in point Z and the hypotenuse in point Y. It the holds that

$$\sum_{EA} XY^2 = \sum_{ED} XY^2 + \sum_{DA}(XZ+ZY)^2 = \sum_{ED} XY^2 + \sum_{DA} XZ^2 + \sum_{DA} ZY^2 + 2\sum_{DA} XZ.ZY.$$

The first and the third sums are basically the same, because it determines the area of two equal triangles. In that way, we dealt with uneven lines and we may consider the remaining two sums to be normal forms, and thus we will in the end obtain the following formula:

$$\sum_{EA} XY^2 = 2\sum_{ED} XY^2 + \frac{1}{4}.$$

Cavalieri stated that the volume of two solids of the same height is the same, if their cuts in the corresponding heights are the same. This principle is nowadays called *Cavalieri principle*. According to this principle, we can sum up the areas and we will get the solid, the sum of squares will thus be a pyramid. We can imagine that the individual cuts are in fact infinitely thin desks. It then holds that

$$\sum_{EA} XY^2 = 8\sum_{ED} XY^2 = 2 \cdot \frac{1}{8} \sum_{EA} XY^2$$

and after simplification we obtain

$$\sum_{EA} XY^2 = \frac{1}{3}.$$

Using contemporary terminology, we could say that for determining the integral, Cavalieri did not use the notion of the limit, but of an *infinitely small quantity*. Cavalieri extended his procedure up to 9, and he thus knew the formula $\sum_0^1 x^n = \frac{1}{n+1}$, while this sum is in fact the integral from 0 to 1. Already Galileo Galilei, Cavalieri's teacher, worked with infinitely small quantities when stating his law of free fall. He knew that speed is directly proportional to the time, i.e. $v = gt$. We are asking about the length of the trajectory the falling object within the time period t. In a certain point in time t_0, the object moves with speed gt_0 and in an infinitely small length of time dt, it crosses the distance gt_0dt, and in the period of time $t - t_0$, the distance $g(t - t_0)dt$. The overall length of the trajectory is $gtdt$, and we would reach the same result if the object moved at both points with constant speed $\frac{gt}{2}$. The length of the trajectory is thus $s = \frac{1}{2}gt^2$. Puritans might feel that this problem does not belong there, but already Archimedes was inspired in many of his mathematical discoveries by physic and the same holds in the 17th century.

Finding the minimum and the maximum of a function was also an important problem. In connection with this, let us mention Fermat's work *Methodus ad Dissquirendam et Maximam et Minimam.* We will demonstrate Fermat's approach to finding the tangent to the curve $y = f(x)$ again on the example of the parabola $y = x^2$ in point $T = [x_0; x_0^2]$. The tangent will intersect the x axis in point P and we need to find its x-co-ordinate. If we increase this value by dx, then we will reach point P_1 on the tangent and the triangles will be similar. If, in addition, dx is very small, we can substitute the curve by a line segment. From the similarity of triangles it follows that

$$PQ = \frac{dx.x_0^2}{(x_0 + dx)^2 - x_0^2}.$$

Fermat then used the not quite accurate trick of putting the square of dx equal to zero, but considering the first power of dx to be non-zero and divided by it both the numerator and denominator, so the x co-ordinate of the intersection of the tangent with the x axis is $\frac{x_0}{2}$ and the slope of the tangent is $k = 2x_0$. If we use contemporary terminology, we could say that Fermat substituted the increment of the function by its differential. Although this approach was not entirely correct, it was used by many scientists and led to correct results. Mathematical puritans did not like this and they protested against this vehemently, but we will describe this later in the sections dealing with crises in mathematics.

Newton allegedly said that he could see further because he was standing on the shoulders of giants. In the case of infinitesimal calculus, we can understand this in such a way that he drew upon the work of all the people who dealt with infinitesimal calculus before him (and we have not mentioned all of them in this book). If we nowadays ask who Newton was, then the majority of those who recognize the name will say that he was an English physicist. This scientist really finished the edifice of classical mechanics and his Principia[2] are as important for physics as Elements are for mathematics. It is less known that this scientist is also to be credited with contributions to infinitesimal calculus and it probably will not surprise us that physics played a role therein.

Newton used his knowledge of analytic geometry and he imagined a curve in a plane (trajectory) as a set of intersections of lines parallel with the coordinate system that move with speeds $\dot{x}$ and $\dot{y}$ and he called these speeds *fluxions*, while the coordinates were called *fluents*. The following must hold:

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dx}.$$

Their proportion is the derivative of y with respect to x. Newton then formulated two basic problems:
1. When we know the dependence of speed on the trajectory of the mass point,

[2]Philosophiae Naturalis Principia Mathematica or Mathematical foundations of natural philosophy, published 1687.

we determine the dependence of speed on time.
2. When we know the dependence of speed on the time, we will determine the dependence of trajectory on the time.

He solved the second problem through seeking a primitive function, i.e. through a procedure inverse to derivation. As Newton then did not yet use the notion of the function, we may suppose that Newton only considered continuous functions. When we consider the narrow connections with the motion of an object, we cannot expect anything else. If a positive function $y = f(x)$ is given on a closed interval $\langle a; b\rangle$ and if we can find such function $y = F(x)$, for which it holds on a certain interval that $F'(x) = f(x)$ in the given interval, then the difference $F(b) - F(a)$ is equal to the area of a formation whose boundaries are the x axis, the lines $x = a$ and $x = b$ and by the graph of the function $y = f(x)$. In other words, we define the so-called Newton integral:

$$(N)\int_a^b f(x)dx = F(b) - F(a).$$

Gottfried Wilhelm Leibniz (1646-1716) chose a different approach. A young man educated in the humanities, he came to Paris in 1672 as a diplomat in the service of the archbishop in Mainz.[3] As a diplomat, he certainly met the Sun King Louis XIV, but it was his meeting with Huygens. It was under the influence of the Dutch scholar that he converted to mathematics and among other things he made himself familiar with Pascal's idea of so called *characteristic triangle* and he developed this thought, used in a special situation, into a coherent theory.

Let the function $y = f(x)$ be given. Let us construct the tangent to this function through point A and let us construct the perpendicular line to this tangent in point A. The perpendicular line, the x axis, and the parallel with the y axis going through point A determine the characteristic triangle APR, where $\mid AP \mid = y$, $\mid PR \mid = m$. This triangle is similar to the triangle ABC, where $dx = \mid AB \mid$, $dy = \mid BC \mid$ are infinitely small. From the similarity of these triangles, we can deduce the equality $mdx = ydy$. We can proceed in this manner in any point and add the results. Leibniz, however, tried to compare the curves $y = x^3$ and $y = x^2$, and he obtained the result $\int x^2dx = \frac{x^3}{3}$. He found the integrals of other curves in the same way.

Both Newton and Leibniz worked independently of each other and their results are equivalent in some ways and different in others. They both discovered the connection between the derivative and the integral and they derived basic formulas, and although they drew upon the work of their predecessors, they enriched their work significantly. If we used some kind of a metaphor, we could say that they finished the basic edifice of the infinitesimal calculus. Newton, however, solved mainly concrete problems and he was often inspired by physics. Leibniz, on the contrary, worked in a more mathematical manner, was looking for general methods and algorithms and he tried to unify the approach to various problems. Newton's symbolism was rather awkward and impractical. Using a dot above a letter before TeX was invented was rather courageous. On the other hand, Leibniz devoted

[3]The archbishop in Mainz was also one of the prince-electors (Kurfürst) of the Roman king.

lots of attention to symbolism he used and we use his symbolism until today. The one thing that was common to both gentlemen was their loose treatment of the infinitely small quantities. We will soon see what came out of that.

Both men also entered a quarrel over the priority of their invention. As we have already stated, they did not invent the theory, but their contribution was central. Newton started to work on these problems earlier then Leibniz, but he published his results with a significant delay. In this sense, Leibniz was a scientist of the 21st century already and he was getting points by publishing his results early. In the second half of 17th century, in addition, there were no scientific journals and to publish the results meant to publish an independent book and their spread beyond the boundaries of the country of origin was an exception. There were no scientific conferences, and the exchange of findings between scientists was through letters, if they corresponded at all. Therefore we can conclude this discussion by saying that they both arrived to their results independently of each other.

4.3 Probability theory

The date of birth of this discipline could be determined to be 1654 m when frequent correspondence on hazard began between Fermat and B. Pascal. Just like Pilatus became part of a credo, a knight Antoine Gombaud de Méré became part of mathematics. This French nobleman liked hazardous games and although we cannot deny that he did have some mathematical knowledge, he could not grapple with some problems, and therefore he asked Pascal for help and Pascal consulted his solutions with Fermat in a letter exchange. Their primary interest was especially the division of a bet among two equal players, who could not finish their game for some reason. Let us suppose that it is as if in the play off part of the tournament, the rules would require four victories (it is not quite clear from the sources what kind of game it was), and when the intermediate result was 2:1 in favour of player A, they had to prematurely end this series. The honour of the players required that the bet be divided and it was also clear that player A should get more, but they could not agree on the ratio.

Fermat realized that the whole series must end after four games. If there are four more games, then there are 16 different sets of results of those games. (Variations of the 4th order with two elements: $\acute{V}_2(4) = 2^4 = 16$.) It is therefore enough to write down all the possibilities of the continuation of the series and determine how many times player 1 would win the bet and the problem is solved. If we do this, we find out that in 11 cases, player A would win and in the remaining 5 cases, player B would win. It is therefore necessary to divide the bet in the ration 11:5 in favour of player A. Fermat correctly realized that it is necessary to take into account all the possibilities, including those that would in reality not happen, because the series would have been won before. There is a certain similarity with penalties in football matches. The rules prescribe fifteen scores or each side, and thus we have to take into account all the possibilities 4^5. A minor remark to end this section: as late as a hundred years later, d'Alembert thought that when tossing two coins, there are only three results possible (heads-heads, tails-tails, and heads-tails). It should not have been difficult for him to take two coins and illustrate the results.

Pascal developed similar ideas, and he expressed in his letters that he was content with the fact that the truth was the same in Toulouse and in Paris. Pascal, however, developed his thoughts even further and he reached a general solution which can be formulated in the following way: Let m victories are lacking for player A to win and player B is lacking n victories to win. Then the number of series that need to be played is at most $m+n-1$. Player A will win the bet if his rival wins at most $n-1$ games. Player B has a chance, if player A only wins $m-1$ times. It is therefore necessary to divide the bet in the following ratio:

$$\sum_{i=0}^{n-1} \binom{m+n-1}{i} : \sum_{j=0}^{m-1} \binom{m+n-1}{j}.$$

It is a pity that this problem does not appear much in textbooks, while the author used it with a lot of success when he was teaching at high school as a motivation problem when introducing theory of probability. More details, and not only about this problem, may be found in [Ma2].

In those times, correspondence belonged to common methods of scientific work, so the problems attacked by Fermat and Pascal were communicated also to other scientists, for example to *Christian Huygens.* [4] This excellent persona of world science contributed also to probability theory by publishing his work *De ratiociniis in ludo aleae.* This work is not very extensive, but is well worth attention. It contains 14 themes that are called Propositio. Let us state the first three of them:

P1: If I expect sum a or sum b, which I can both obtain with the same effort, then the value of my expectation is $\frac{a+b}{2}$.

P2: If I expect sum a, sum b or sum c, which I can all obtain with the same effort, then the value of my expectation is $\frac{a+b+c}{3}$.

P3: . If the number of cases in which I expect sum a is p and the number of cases in which I expect sum b is q, and if I assume that all cases can happen equally easily, then the value of my expectation is $\frac{pa+qb}{p+q}$.

In these three definitions, expected value is defined for the first time, although Huygens does not state this explicitly. As the book deals with games, he uses the term expected winning. Similarly, he does not use the term probability and instead says that all the results can be obtained equally easily.

Propositiones 4-9 solve the problem of dividing a bet. Huygens starts with simple examples and proceeds to the more complex ones and he also adds players. We will only state the full version of Propositio 9, as in that one, Huygens states the general method for solving the problem, by which he reached a level higher than Fermat or Pascal who only solved concrete problems.

P9. In order to calculate the share of each player in a game with an arbitrary number of players, from whom some lack more and other lack less victories, we must find out what belongs to the player whose share should be determined when he himself or somebody else wins the next game. By adding the parts obtained in this

[4] Christian Huygens (1629-1695), Dutch scientist who lived in Paris for a part of his life. He discovered, among other things, the rings of Saturn and formulated wave theory of light.

way and dividing it by the number of players, we obtain the share of that particular player.

A table presenting the solution of the problem for 17 situations that can happen in a game of three players is attached to this Propositio. The remaining Propositia concern the game of dice. At the end of the book, we find five unsolved tasks (Problemata). We refer those who are interested in more detail to book [Ma2].

When dealing with mathematics of 17th and 18th centuries, we can be sure that we will encounter the name Bernoulli. Probability is no exception. We need to mention especially Jacob I (1654-1705), the author of *Ars Conjectandi.* This book remained incomplete, it was published only in 1713 by Jacob's son Nicolaus I. In the first part of the book, we will find the full text of Huygens's De raciociniis in ludo aleae including detailed commentary. Second part is basically a course in combinatorics and the third part draws on to the second, as combinatory problems with the theme of games. In the last, unfortunately unfinished, part, we can also find the first formulation and proof of the law of large numbers, which is probably Jacob Bernoulli's greatest contribution to probability theory.

High school students who have probability and statistics included in their school curriculum will probably meet the name of Bernoulli in connection with binomial distribution. Let us perform n independent attempts, while the probability that the phenomenon A occurs in one of the attempts is p and it is the same for all the attempts. Then the probability that the phenomenon A occurs in this series exactly k times is given by the formula

$$p_k(A) = \binom{n}{k} p^k (1-p)^{n-k}.$$

This arrangement is sometimes called Bernoulli's scheme or Bernoulli sequence of independent attempts and it provides a wide opportunity to present problems in probability. In any case, the following problem should be presented to the students:

The test consist of 20 questions, and for each questions, four possible answers are given, of which only one is correct. A student did not prepare for the test and fills out the answers at random. What is the probability that he will pass the test, if he needs to answer 15 questions correctly for that?

The probability that a high school student will meet the name of Bernoulli in connection with another result in probability theory is not very high, which is definitely a shame. Jacob's nephew Daniel published the following problem including its solution and as it happened in St Petersburg, it is known as the *St Petersburg paradox.*[5] The author of this book never forgot to mention it to students, it was very well received, but none of his students ever solved it. What, then, is St Petersburg paradox?

Two players, of whom one is a banker,[6] play the following game: the banker tosses a coin and if it turns out to be heads, the game ends and the banker will pay his rival 2 crowns. In the opposite case, the game continues with another toss and

[5] This problem was only published in the 18th century, but as we will not be mentioning probability theory in the following text again, we decided to present it here.

[6] Here, we do not have in mind the owner of a financial establishment, but the person accepting bets and giving away the winnings.

it the result is heads, the game ends, but now the banker has to pay 4 crowns. The game goes on until the banker does not succeed in tossing a heads. If this happens in the n-th attempt, the banker will pay his rival 2^n crowns. The question is, how many crowns the player must put into the game so that the game is just, i.e. so that the victory or loss of one of the players was only a matter of chance.

As has already been said, when the author presented the task to his students, none of them solved it, although the solution is not difficult and the necessary formula had been taught before. The expected value of the winning is namely

$$E = \sum_{i=1}^{i=n} p_i E_i = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \cdots$$

In order for the game to be just, the player would have to start by paying infinitely many crowns. This was a great surprise for the students, especially when they tried this game before the calculation. Calling it a paradox is thus fully justified.

If the students have already been taught about geometric series, the teacher can return to this problem again. The paradox consists in the fact that the banker has to pay any amount of money the player wins. Geometric sequence is tricky in this sense, as one Indian maharaja who could not pay the reward he promised to the inventor of chess could confirm, as well as those who have trusted a fraud in pyramid scheme type of game. We therefore adapt the game slightly by determining that the highest winning to be paid out is 64 crowns. The expected value of the winning will thus be

$$E = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \frac{1}{16} \cdot 16 + \frac{1}{32} \cdot 32 + 64 \left(\frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \cdots \right),$$

i.e. roughly 5.03 crowns. Even in this case, however, we need to take into account all the possibilities, even if the game would in reality end with the fifth attempt.

Z uvedeného textu by se zdálo, že i tak serózní věda jako je matematika nebyla imunní vůči gamblerství a bezuzdně mu propadla. Jenže tomu tak není. Teorie pravděpodobnosti našla záhy své místo v pojistné matematice, neboť na základě statistických údajů (úmrtnostních tabulek), lze stanovit pravděpodobnost, že se daná osoba dožije věku v. Spojení teorie pravděpodobnosti a statistiky je přirozené, na pravděpodobnosti je založena i moderní fyzika, přesněji řečeno kvantová mechanika. Závěrem povídání o pravděpodobnosti přineseme několik zajímavostí z našich luhů a hájů. Jedním z prvních načich matematiků, kteří se zabývali touto disciplínou, byl Václav Šimerka[7], který se snažil pomocí pravděpodobnosti řešit některé filozofické otázky. [Sm2].

4.4 Rudolphine Prague

Also this chapter will be closed with a few remarks about mathematics in the lands of the Bohemian crown. This time, this section will not be short and,

[7] Václav Šimerka (1819–1887), kněz, učitel a matematik.

in particular, it will not concern only matters of regional interest. In 1576, the emperor Maxmilian II died and the heir to the throne was his oldest son, Rudolph, second of that name among Habsburg kings. Bohemian noblemen then performed a daring escapade and succeeded in convincing the young emperor to move to Prague. Prague agglomeration thus became the centre of the central-European union of states and its fame again reached the stars again after more than two centuries. The emperor had somewhat weird habits, but arts and sciences were his passion and Prague became a favourite destination of scientists as well as "scientists". We will not pay attention to the latter ones, although, according to the well-known movie directed by Frič, they indeed invented some good things like glue, floor polish, and plum brandy, for example. We will rather note the real scientists, both local and from abroad, who were attracted by Rudolph's generosity, although there often was a significant difference between the promised and received salary.

In the first place, we need to mention the astronomer, botanist, and personal physician of the emperor, *Hagecius* (also known as *Tadeáš Hájek of Hájek* or *Thaddaeus Hagecius ab Hayek*, 1525-1600). It was Hagecius who invited the Danish astronomer *Tycho Brahe(1546-1601)* to Prague. Tycho Brahe received the castle in Benátky nad Jizerou, where he transferred his astronomical observatory. Brahe was an excellent observer and he constructed and improved many astronomical instruments, but he did not ask the emperor for lenses, since the telescope was only invented several years after his death. When Brahe died, he left lots of material about the movement of celestial bodies that he did not manage to process. This task was taken up by the court mathematician of Rudolph, *Johnanes Kepler (1571-1630)*. Kepler focused in particular on the movement of Mars and the more he looked into the data, the more he came to realize that it will be necessary to leave the thousand years old idea of circle as the most perfect of all curves and that therefore all celestial bodies must move along circular trajectories. Thanks to this revolutionary thought, it was possible to put the heliocentric theory of Copernicus in accord with the movement of the planets in reality. Ptolemy's geocentric system, although it was very complex, with all the epicycles that had to be added all the time, was namely giving better computational results. Kepler's *Astronomia nova* was very important for further developments in astronomy, as his laws described the movement of big celestial bodies. Kepler succeeded in describing the movement of the objects in the sky, but he could not explain why it is so. That was only achieved by Newton, who discovered the general law of gravitation.

Let us shortly remind ourselves of his laws. The first one says that the planets move on elliptical trajectories in the centre of which is the Sun. The second law says that a line joining a planet and the Sun sweeps out equal areas during equal intervals of time (areal velocity is the same). Thanks to this law, the summer on the northern hemisphere is by several days longer than the summer on the southern hemisphere. When there is summer in our country, the Sun is in aphelion (farthest from the Sun) and it therefore moves more slowly. The Russians sent their telecommunications satellites on trajectories with big eccentricity and these then stayed for a long time above their territory. Kepler later added also another law, saying that the square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit. To end this passage on Kepler, we need to correct the information given in the famous film directed by Frič. It was not Tycho Brahe, but Kepler who was the professor who asked the emperor for a lens and his request was fulfilled. Kepler already used a telescope and, contrary to Galilei, his eyepiece had a lens.

Kepler also belonged to the pioneers in using algorithms. His tables had been assembled by his assistant and a clockmaker from Switzerland, *Joost Bűrgi (1552–1632)*

Chapter 5

From 18th century till now

The history of humankind in this period is full of revolutionary events and life changed significantly in it. Just as technology was developing fast, mathematics also had to be developing fast. We will, paradoxically, shorten our account of this period, because the results are glamorous, but they exceed the framework of mathematics taught at elementary schools and high schools. Those interested in the development of mathematics in this period are thus advised to look into other sources.

5.1 The enlightened century

This section is named after the movement that characterizes this century. However, as we are writing about the queen of the sciences, the title Euler's century would be more appropriate, after the mathematician *Leonhard Euler (1707-1783).* As his name was mentioned in other parts of this book, we will not deal with his personality in detail. Bernoulli family, in which the mathematical genes were inherited so consequently that some of the family members have to be referred to by numbers next to their names, as with kings, also has Swiss roots. Brothers Johann (1667-1748) and Jacob (1654-1705) devoted their attention to analysis, in which they achieved substantial results, which are nowadays part of the main body of knowledge called mathematical analysis. The contribution of this family to probability theory has already been mentioned. Bernoulli equality, known from hydrodynamics, which is taught in physics, is the discovery of Johann's son Daniel (1700-1784).

In France, which is not far from Switzerland, many excellent mathematicians were active in this period. *Jean Le Rond d'Alembert (1717-1783)*, from 1754 permanent secretary of the French Academy of the Sciences, was also a leading figure among the encyclopaedists, especially for mathematical part of it. Together with Daniel Bernoulli, he is the founder of the theory of differential equations. He is also one of successful physicists (hydrodynamics, aerodynamics, three body problem). His mechanical principle, reducing dynamics to solving static problems, is well known. He introduced the notion of the limit. He also devoted some attention to probability theory, but was not very successful. We know his "paradox", in which he mistakenly determined the probabilities for tossing two coins. He said that the probability that the heads falls at least once is $\frac{2}{3}$, because he considered the possibilities HH, HT, TT instead of the correct HH, HT, TH, TT. The correct result therefore is $\frac{3}{4}$.

Joseph Louis Lagrange (1736-1813) was born in Turin and was of French-Italian origin. He worked in Turin, Berlin, and Paris. To begin with, he devoted his time to the calculus of variations and he discovered many original results in this field. Apart from that, he also ordered and compiled historical material, which was typical for him also in other fields of his interest. He then applied his discoveries in mechanics, for example, he provided the first particular solution of the three body problem. He summed up his results in his book *Mécanique analytique*, in which he united the different principles of statics and dynamics. In the infinitesimal calculus, he introduced an algebraic approach when starting from the development of functions in Taylor series. Let us mention, as a curiosity, that the symbols for derivation that we use nowadays are the work of this particular man. During French revolution, he was the president of the measures and weights committee.

Pierre Simon Laplace (1749-1827) published two big pieces of work: *Théorie analytique des probabilités* and *Mécanique céleste*. In the second book he summarized the contemporary knowledge of mechanics. We have already spoken about Napoleon's relation to mathematics, so we cannot be surprised that he was also interested in this big volume and he wondered why it did not mention God anywhere. Laplace answered: "Sir, I did not need this hypothesis. " We however, have to remind the reader that Laplace had good relations also with the Bourbons: his short episode as a minister in the times of Napoleon's rule was forgiven, and in addition, he did not prove himself to be a good minister. The worse politician, the better scientist he was and his work inspired many other scientific colleagues. Thanks to him, the name of Bayes was not forgotten, and thus we today use the formula named after this English priest for calculating probability a posteriori.

In the Czech lands, mathematics was mainly devoted to applications. A typical example of this is the work *Solid beginning of the art of mathematics* by Wenceslas Joseph Weselý, a land miller and geometer, in which we can find examples of practical surveying including trigonometry. Other books were devoted to practical or merchant mathematics, calculating the interests, and other problems. Only in the second half of the 18th century did Czech science and thus also mathematics start to make up for the delay. *Joseph Stepling (1716-1778)* published his book *Exercitationes geometrico-analyticae* in 1751, which is devoted to the integration of certain analytic functions and which raised some interest and was soon published two more times. In 1765, he then compiled a work on differential calculus, in which he also tried to extend Euler's thoughts. He is also to be credited with building the observatory in Klementinum in Prague, which can boast with the longest sequence of meteorological observations in Europe, having started them in 1775. In 1780, Newton's Principia was published, to which *Jan Tesánek (1728-1788)* wrote an introduction. In this work, he tried to make the notion of limit more exact and thus to link this theory to d'Alemberts. He was also one of the first number theorists in the Czech lands and he tried to solved the so-called Pellé's equation $du^2 + 1 = v^2$. *František Josef Gerstner (1756-1832)* is probably the best known name from this period: he designed the known railway with carriages drawn by horses from Budweis to Linz. That proves that his domain was rather application of mathematics in practice. Around 1770, the Learned Society was founded, which later developed into Royal Bohemian society of arts and sciences.

5.2 The century of steam

This is the traditional name for the 19th century, and it probably should signify that there was a significant development in the industry and in agriculture, and that the symbol and motor of these changes was the steam engine. Science, including mathematics, also had to comply with the new situation. The age of the titans who mastered various and often very different branches of science slowly ends and the time of specialization comes, also in mathematics. The practice of mathematics moves to schools, and mathematicians, apart from doing research, devote more and more of their time to teaching mathematics. This century also brought new branches of mathematics.

One of the last scientists of the old age was *Carl Friedrich Johann Gauss (1777–1855).*

Geometry witnessed a major development. *Gaspar Monge (1746-1818)* taught at the military academy in Méziers on the construction of fortresses and thanks to this, he began to develop descriptive geometry as a special field. When studying space curves and surfaces, he started using infinitesimal calculus, by which he established another field - projective geometry. His thoughts were further developed by *Victor Poncelet (1788-1867)* in his *Traité des propriétes projectives des figures*, in which all the basic notions of this field are included.

The birth of new fields, however, was nothing compared to what Euclid caused and what surfaced exactly in this century. Euclid formulated his postulates in geometry in his first book and while the first four are formulated in a very simple form, the fourth one is rather clumsy and talkative; you may judge it for yourselves:

1. I can draw a straight line from any point to any other point.
2. And [I can] prolong the bounded straight line without a break.
3. And [I can] draw a circle with any centre and with any radius.
4. And all right angles are mutually equivalent
5. And should a line fall on two other lines in such a way that the inner angles along one side added together are smaller than two right angles, then if we prolong such lines unboundedly, they will meet on the side where the angles are smaller than two right angles.

This disparity and difference in formulation was noticed already by Ancient scientists and thus they had the idea of proving this postulate, whether it is not a mathematical theorem.[1] Ptolemy, Násiruddín Túsí, Lambert , and Legendre. The first of them lived in the Antiquity, the second in the Middle Ages, the last two in the 18th century, and thus we can see that the problem troubled mathematicians practically continually for over two millennia. Neither those whom we named, nor others who tried to tackle the problem, however, found the proof. Not even Gauss was successful, but he came up with the idea that this postulate is independent of the others and that therefore it is possible to leave it out or to substitute it with a different one. However, this thought felt too daring to him and Gauss kept it for himself, but luckily, he was not the only one who developed thoughts in this way and these other two gentlemen had the courage to publish their idea. The first one was the Hungarian *János Bolyai (1802-1860)* and the second the Russian *Nikolaj Ivanovich Lobachevsky (1793-1856).*

[1] Later, it was found out that the fourth postulate could also be proved, but this fact did not cause any hassle.

Chapter 6

Three crises in the history of mathematics

6.1 The first crisis

We have already suggested some causes for this crisis in the section on Pythagoreans, but the discovery of irrational numbers was not its cause. We can mention *Zeno of Elea (490-430 BC)* and his famous aporias. Originally, there were around 40 of them, but only 9 were preserved until today and it is questionable whether they are original or whether they have been distorted during the copying of the texts. The most famous aporia is the one about Achilles and the tortoise, which we do not even have to repeat here. Let us mention also the one about the flying arrow, the one that denies the existence of movement. If, Zeno says, we look at a flying arrow, we only see it because it at that very point stays in the same place. Motion therefore consists of a number of motionless moments, which is not possible. In the aporia on dichotomy, he then states that we cannot move from point A to point B, because we first have to travel half of this distance, and then half of this half, etc.

Even a small boy, however, can surpass a tortoise, let alone Achilles, and the archery masters like William Tell, Robin Hood, or the Native American Indians proves that the arrow moves. In the same vein, everybody can travel from one point to another, although sometimes we might prefer this not to be possible. These facts, however, could not confuse Zeno and his argument ran roughly as follows: Yes, it is true that we can see movement, but if we want to understand what the eyes are seeing, we have to approach movement in the way I had suggested. As I have shown, we cannot think of movement, because movement is only proper for the changing world of the senses, but alien to the real being. In his aporias, Zeno showed the opposite of the continuous and discrete movement, the movement and its reflection in our minds, or, in other words, the opposition between the cognition through senses and through reasoning. After all, we meet this paradox in mathematics quite often, with non-Euclidean geometry being a beautiful example.

The Greeks have dealt with the problem of irrational numbers by geometrizing mathematics. We can imagine the square root of two as the diagonal in a square with the side equal to one. We can see it clearly in books 7—9 of Elements, where Euclid understands numbers to be line segments.

6.2 The second crisis

The origins of the second crisis also lie in Ancient Greece, but it only fully grew in the 18th century. If we simplify it, we could say that this crisis was caused by the infinitesimal calculus, or, more accurately, the way how this discipline established itself. Let us remind ourselves about the way how Archimedes performed squaring of the parabola. In plain language, he filled the shape with triangles until he considered that calculated value accurate enough. Archimedes, however, was a rather solitary scientist, but in 17th century, more and more scientists dealt with problems of this kind. From the examples given it is obvious that they approached the problems rather loosely and they did not pay much attention to the details. If we, for example, look at Newton's way of deriving the formula $(x^n)' = nx^{n-1}$, it is hardly understandable from today's point of view. Why not, if it is convenient, could we not put an infinitely small quantity o equal to zero and then neglect it? On the other hand, at other times, we need to divide by this small quantity and in that case, it is inconvenient that it is equal to zero, so we say it is not equal to zero and divide by it as we please.

We may boldly liken the establishment of infinitesimal calculus to the times when people began to settle in the Wild West. A discovery after discovery was being made, without the mathematicians reflecting much on the way how they arrived to that result, what counted was that it gave results that corresponded well with the reality and helped to grasp it, especially in physics. The procedure for deriving more and more discoveries was rather formal, and the scientists of that time had imagination and especially a good nose for reaching good results. The author apologies for this familiar expression, but he has not found a better fitting expression in standard language. Euler was a real master in this sense, and we will demonstrate his skill on two examples - in the first case, he arrived to the correct result, but in the second one, he was wrong, but that was a very rare exception in his case, and it only confirmed the rule that Euler's work is part of the pillars (not only) of analysis.

When teaching complex numbers in high schools, we calculate the powers of a complex number with the aid of Moivre theorem and the binomial theorem. Euler proceeded in the same way when he considered infinitely large n, so that $\varepsilon = \frac{x}{n}$ was infinitely small. It holds that

$$(\cos\varepsilon + i\sin\varepsilon)^n = \cos n\varepsilon + i\sin n\varepsilon = \sum_{k=0}^{n}(i)^k\binom{n}{k}\sin^k\varepsilon\cos^{n-k}\varepsilon.$$

If we for example compare imaginary parts, we obtain

$$\sin x = \binom{n}{1}\sin\varepsilon\cos^{n-1}\varepsilon - \binom{n}{3}\sin^3\varepsilon\cos^{n-3}\varepsilon + \dots .$$

As ε is infinitely small, then $\cos\varepsilon = 1$ and $\sin\varepsilon = \varepsilon$. As n is an infinitely small number, $n = n - 1 = n - 2 = \ldots$. The equality can thus be rewritten in the following form:

$$\sin x = \frac{n\varepsilon}{1} - \frac{n^3\varepsilon^3}{!} + \ldots = \frac{x}{1} - \frac{x^3}{3!} + \ldots$$

which is the correct expansion of the sine function into power series. We will obtain the expansion of the cosine function in the same way.

Euler thought that each infinite sum will be a result of a "correct function". Therefore he thought that

$$1 - 1 + 1 - 1 + \ldots = \frac{1}{2},$$

since it suffices to substitute in the expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

number 1 for x. Guido Grandi, a professor from Pisa, arrived at the same conclusion, namely by paring conveniently $(1-1)+(1-1)+\ldots = 0$ and $1-(1-1)+(1-1)+\ldots = 1$ and using the statistical consideration known from jokes and comedy scenes.[1]

It is thus not surprising that some mathematicians did not like the methods described above and that they started criticizing them. Thus *Bernard Nieuwentijdt (1654-1718)* criticized Leibniz's approach and he in particular said that higher degree derivatives do not have any sense. The bishop *George Berkeley (1685-1753)* then attacked Newton even more directly. In 1734, he published the book *The Analyst*, in which he uses similar arguments as those that we have given earlier. It was not really possible to deny the correctness of the results, but Berkeley thought that the correct results were obtained only because two mistakes basically compensated each other. He called infinitesimal quantities ghosts of deceased quantities and he said that who will believe in second or third fluxion, or the second or third derivative, cannot deny any feature of a deity. Berkeley's tractate namely has the subtitle. *A Discourse addressed to an Infidel Mathematician*, who was nobody else but Newton's pupil *Edmund Halley (1656-1742)*. He is the same man who used Kepler's and Newton's laws to calculate the trajectory of one significant comet, which he thus ranked among the satellites of the Sun. The comet, bearing his name, keeps to the timetable he had prescribed for her, unlike trains and other public transport, and every 78 years, it shines in the sky. The story has it that the infidel Halley lured away one soul from the bishop, which made the religious man angry and provoked him to write the book we have just mentioned.

This story may only be well thought-out fable, but it captures the core of the problem. Although mathematical tools were created in a strange way, it was not possible to deny that they helped in an excellent way the physicists and the astronomers solve problems posed before them Things have reached such a state

[1] When I eat one chicken and my friend eats nothing, we have both eaten half of a chicken. Grandi was not that down-to-earth and gave an example of two brothers who take turns and hold on to the diamond inherited from their father alternately for a year at a time.

that when Laplace presented his *Mécanique céleste* to Napoleon, he could answer his question as to where God is, "Sir, I did not need that hypothesis". It was therefore not by chance that the opponents of infinitesimal calculus were recruited especially among idealists. Berkeley namely was not only a parish priest, but also a well-known idealist philosopher. It was therefore obvious that something needs to be done in this matter.

The first problem was the function itself. Albeit founding fathers worked with functions, a function to their understanding was some dependence expressed by a formula and functions were of course continuous. The first definition of a function in the contemporary sense comes from Johann Bernoulli from 1718: *A quantity composed in an arbitrary way from a variable an constants is called a function of the variable.* Euler proceeded in the same way, although more accurately: *Analytical expression composed in an arbitrary way from variable and constant quantities is the function of the variable.* Seven years later, in 1755, he gave the following definition: *When some quantities depend on others in such a way that when we change those latter ones, the former ones change as well, we say that the former are a function of the latter.* In 1822, Fourier wrote: *Generally, function f(x) is a sequence of values of which each is arbitrary; we do not assume that these values are subject to some law, they follow one another in an arbitrary way.* Fourier also left the idea that a function must be continuous, although he only considered a finite number of points of discontinuity. In 1829, Dirichlet states the following: *y is a function of x, if only one single value of y from the interval given exists for each value of x.* He goes on to add that it is not of importance whether some formula for this exists or not. His definition is used until today. He gives an example of a function where $y = 1$ when x is rational and $y = 0$ when x is irrational.

The real number itself was another problem. On the other hand, the sets of natural, integer, and rational numbers can be defined rather easily, be it intuitively or with the help of axioms, as *Giuseppe Peano (1858-1932)* did. He defined the natural number through the following five axioms: 1. $1 \in N$
2. Number 1 is not a successor a^+ of any number $a \in N$.
3. If $a \in N$, then also $a^+ \in N$.
4. If $a^+ = b^+$, then $a = b$.
5. Let $S \subset N$, $1 \in S$ and let for each $a \in S$ it holds that $a^+ \in S$. Then $S = N$. (This is the axiom of mathematical induction.)

With these axioms, we can then define the sum and product of natural numbers, integers as $a - b$, and rational numbers as $\frac{a}{b}$, i.e. as ordered pairs of natural numbers. It holds that the ordered pairs (a, b) and (c, d) determine the same integer if and only if $a + d = b + c$ and these two ordered pairs determine the same rational numbers if and only if $ad = bc$. This approach might look rather complicated to the reader, but the simple definition uttered by the German mathematician *Leopold Kronecker (1823-1891)* was not accepted by the mathematical community at all.[2]

The construction of real numbers from rational numbers is not easy, but mathematicians nevertheless succeeded in it around 1870. In this connection, we in-

[2]This man, who also had a strong influence also M. Lerch, claimed that natural numbers are from God and everything else is human invention.

troduce the names of *Charles Méray (1835–1911)*, *Georg Cantor (1845–1918)* and *Richard Dedekind (1831–1916)*.

Štefan Znám made a jesting comparison in his book [Zn] by comparing the chateau of infinitesimal calculus, constructed over two centuries, to the house of Baba Yaga. He did not mean its size or its grandeur, but the fact that this chateau, like Baba Yaga's house, stood on a chicken leg. It was therefore necessary to construct firm fundaments for this building. At this point, we will introduce the names of *Bernard Bolzano (1781-1848)*, *Augustin Louis Cauchy (1789-1857)*, and *Karl Weierstrass (1815-1897)*. The first two names are connected with Prague. Bolzano lived in Prague and worked there as a professor. Cauchy then several times lived in this town with the deposed king Charles X, because he was a teacher of his children. Bolzano's works were, however, not known to the public (if he published his thoughts at all). Cauchy's works, on the contrary, were widely known, especially his masterpiece *Course d'Analyse* published in 1821. Cauchy correctly defined elementary notions as continuity (Bolzano could also do that) and limit, he also uses the notion of infinitely small quantity, which he further specifies later. Cauchy (mistakenly) thought that each continuous function has a derivative except for a finite number of points. Bolzano was the first one to construct a continuous function which did not have a derivative in any point, but he never published this discovery. The firm foundations for mathematical analysis were completed by the last of the three, who introduced the ϵ-δ symbolism used until today. His definition of the limit created difficulties for generations of students, but it had beneficial influence on the infinitesimal calculus. Thanks to this approach, it was possible to remove intuition from analysis and start with proving the theorems correctly. This process is sometime called the *arithmetization of analysis*, when the previous dynamical approach was substituted for by static approach. From the point of view of mathematics, this approach is faultless, but for teaching at high schools, it is advisable and follows from the author's experience to start explaining the derivative as Newton did. Dynamic approach is in compliance with practice in technology, when planning a railway viaduct, only the weight of the passing trains is taken into account. The fact that a hamster can also appear on the bridge at the same time with the train is overlooked, although this cute animal also loads the bridge.

6.3 The third crisis

This crisis also has its origin in Ancient Greece when Epamenides, the citizen of Crete, disgusted by the dishonest behaviour of his fellow citizens, uttered the famous sentence that all Cretans are liars. This has really happened, and was noted by Eubolides of Miletus, and we will return to this later. This statement seems not to have anything in common with mathematics, since it concerns logic.

The third crisis is due to set theory and the notion of infinity. Even Ancient Greek mathematicians had to grapple with this notion, but they always understood it in the sense of *potential infinity*, i.e. in the sense that some set does not have limitation. The second axiom of Euclid states that a line segment (a line) can be arbitrarily prolonged beyond its endpoints, a statement in Book X then states that there are more primes than any given number. In fact, until the 19th century, nobody came up with the idea of thinking about infinity in a different way. The first one to think of sets differently was the already mentioned Bolzano in his posthumously (1851) published book *Paradoxien des Unendlichen.* Bolzano namely thought of trying to watch the situation from above. In the work just mentioned, he a.o. says: *In order to think of a whole (an infinite whole) composed of objects a, b, c, etc., we need not have an image of each of them in our minds. We can think about the set of all the inhabitants of Prague as a whole without having an image of each of the inhabitants of the town.* Cantor was another one who looked at sets as a whole, but before we state what he found out, let us present a short diversion.

Let there be two finite sets and we should find out which of them has more elements. One of the possibilities is that we count the number of the elements of the first and of the second set and compare the results. Can also a person who cannot count solve this task? The answer is yes, it the person thinks of pairing the objects from the two sets. Using this approach, the greater set is the one in which some elements are left over. If nothing is left over in neither of the two sets, then the number of the two sets is the same, although we do not know what the number is. A mathematician would say that two sets have the same number of elements when a bijection exists between them. If we use this approach for the set of natural numbers, strange things start happening. It is not very well known that one of the first people arriving to this conclusion was Hugo Clever, who worked as a receptionist in Brno Grandhotel and faithful to his name, it had at that time an infinite number of rooms. One day, when there was a downpour of rain outside, a tired old man came to the hotel and asked for a room. Mister Clever felt sorry for this man and although the hotel was full up, he found a room for this guest through an organisational arrangement. We already suspect that he moved the guest from room number 1 to room number 2, guest from room number 2 to room number 3, etc., and through this operation, room number 1 was free and no guest was left without a room. Mister Mazany did it for free for the tired old man, but in other cases, he only did it for a bribe, through which he earned considerable money, especially in the times of fairs and Grand Prix competitions, because the rumour about the clever receptionist who can accommodate n guests even when the hotel is full up spread fast. When Mister Clever managed to accommodate, for an adequate bribe, even a train with infinitely many tourists (he moved each guest to the room whose number was a double of their original room, through which he obtained infinitely many rooms with odd numbers), he stopped this activity and decided to spend his money in the most popular holiday resorts.

Cantor was concerned with the problem of uniqueness of Fourier series, and he found out that the condition can be weakened and infinitely many exceptions, in some sense, may be allowed, without any detrimental effects to the uniqueness. Cantor therefore introduced (intuitively) the notion of set, one-to-one mapping between sets, and the notion cardinality. He stated that the set of natural numbers is countable and he denoted its cardinality as $\aleph_0$. He proved that the set of rational numbers was countable; similarly, that the set of integers is also countable and on the other hand, that also the set of squares of natural numbers is countable. As not every natural number is a square of another natural number and since at the same time each square of a natural number is a natural number, the two sets are in the relation of inclusion. On the other hand, there is a one-to-one mapping between them, and they thus have the same cardinality (same number of elements). Cantor thus denied Euclid's common axiom 8, which states that the whole is larger than its part. Cantor also proved that the set of real numbers is not countable, and that there are thus at least two different infinities. Cardinality of the set of real numbers is called continuum and is denoted by $\aleph$. For real numbers, it also holds that a whole need not be greater than its part, as the cardinality of any interval of real numbers is again continuum. Finding a one-to-one mapping between two intervals is no problem at all.

At the same time, contemporary with the rise of the new theory, difficulties began to arise, or rather problems that could not be solved and that began to be called paradoxes or antinomies of set theory. The first of these antinomies, known already to Cantor and published by Cesare Burali-Forti in 1897, can be formulated in the following way: Ordinal number of a well-ordered set of all ordinal numbers is larger than all ordinal numbers (an ordinal number larger than itself exists). Another antinomy can be formulated in the following way: Let M be a set of all sets that are not their own elements. Then, however, it holds both that $M \in M$ and $M \notin M$. *Jules Richard (1862-1956)* drew attention to another paradox in 1905. His paradox consists in this: Any natural number can be described by a sentence in some language. If we choose Czech, then that sentence is represented by a certain sequence of letters of the Czech alphabet. There are less than 50 letters of Czech alphabet, and thus there are less than 50^{100} Czech sentences containing less than 100 letters. There are, however, more natural numbers than this number and there is the least one of those, which we will denote by n_m; n_m is the least number that cannot be described by a sentence with less than 100 letters. But that is exactly what we have done now.

There are, however, similar problems in logic. Epamenides was definitely convinced that he was the only truth-speaking Cretan. But according to his statement, all Cretans are liars, and thus he is also a liar. In that case it is not true that all Cretans are liar. But if the one truth-speaking is Epamenides, then he uttered a truthful expression. The author also likes to remember the time when he as a cute kid passed the Sunday afternoons in barbers' shop, because all man had to be shaved on Sunday. It can thus be said that there were two kinds of men, in the first one, there were those who shaved themselves, and in the second one, those who were shaved by the barber. However, which group does barber belong to?

Mathematicians (and also philosophers) were looking for a way out of the crisis, but they did not act as a unanimous group and several directions of dealing with the crisis emerged. The *intuitionists* rejected the actual infinity and favoured intuition. Among other things, they rejected the principle of tertium non datur (for each proposition, either V or $\neg V$ holds). *Luitzen Egbertus Brouwer (1881-1966)* was the founder of intuitionism. As was suggested above, some antinomies from set theory have their analogies in logic. *Logicism* regarded logic to be the foundation of mathematics, and of course only the modern logic, without impredicativeness (a Cretan cannot speak about Cretans). *Bertrand Russell (1872-1970)* and *Alfred North Whitehead (1861-1947)* are the two most important representatives of this direction. *Formalism* is best represented by *David Hilbert (1862-1943)*, who refused to be thrown out of the paradise created for mathematics by Cantor. The main idea of this direction is the establishment of a rigorous formal system like the one of Euclid, i.e. to establish a complete and consistent system of axioms, from which any proposition could be proved with the aid of definitions. This mathematics was taught at schools and it seems that no other direction is possible any more. However, then *Kurt Gödel* should not have been born in Brno, since he stated and proved the following proposition: *Let S be a complete and consistent system of axioms. Then an undecidable proposition exists in this system.*

As contemporary mathematicians agree, we have not overcome the third crisis yet. Each of the mentioned directions (and also those that have not been mentioned here) certainly contributed to the development of mathematics, but it did not solve the crisis. Is the situation in mathematics similar to the situation in physics at the end of 19th century when it seemed that after a couple of last problems are solved (photoelectric phenomenon, radiation of an absolutely black object), the physicists would no longer have any work to do? Will a mathematical Einstein come and tear contemporary mathematics to thrash? But just like Newtonian physics did not cease to exist and continues to serve well, nor will the mathematics of Pythagoras, Euclid, Archimedes, Fermat, Newton, Euler and other giants mentioned in the previous sections cease to exist. Whether or not mathematicians come up with anything in future, the author thinks that the bricklayers will keep demarcating the right angle with the aid of the triangle with sides equal to 3, 4, and 5. Mathematics will certainly not lose it position and anyone who deals with it should know its history, to which this book strives to contribute.

Part II

Biographies

In this part, we will introduce short biographies of a few important mathematicians whose names we encounter when teaching mathematics and elementary schools and high schools. The selection is necessarily subjective, but the length of this text does not allow us to introduce all the people who contributed to the shape of mathematics taught in schools today. For scientists from the antiquity especially, we only have a few biographical data that could be verified in other sources, and therefore we sometimes mention facts that are perhaps not true, but only good fables, because we think that even such things belong to teaching, since they help mathematics look more human.

Chapter 7

Foreign mathematicians

7.1 Archimedes of Syracuse

Archimedes was regarded the most significant scientist of the Antiquity and we think that this was well deserved. This personality of Hellenic science namely achieved the thing that is demanded so often today, namely connecting theory and practice. Using today's terminology, we could say that Archimedes excelled in both pure and applied research and that he mastered several branches of science.

He was born in 280 BC in Syracuse in Sicily. This town, in contrast to what it is today, was then one of the most important centres of Ancient Greek civilization, both in the classical and Hellenic period. In the times of Archimedes, the centre of education was in Alexandria. Archimedes also maintained scientific ties with his colleagues in Mouseion.

As has already been said, he could connect theory and practice. His most famous discovery is the one about buoyancy forces exerted on an object partially submerged in fluid; this principle is named after him and can be formulated as follows: *Any object immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the submerged part of the object.* This discovery is usually connected with the task Archimedes was given by the tyrant king Hiero II. This king had a new gold crown made and he suspected the goldsmith of cheating. Therefore he asked Archimedes to solve this problem. However, Archimedes could not solve the problem, until he in a bath realised that he could solve the problem by weighing the crown in water and on dry land and compare the results with the weight of gold of the same weight. According to the legend, he was so enthusiastic about his discovery that he ran out of the bath and crying heureka (I found it), ran to announce the solution to his king.

When Hannibal invaded Italy, the town of Syracuse joined him. In 215 BC, they were besieged by the Roman general Marcellus. Although the Romans by far outnumbered them, the town of Syracuse resisted the siege for three years and apart from the bravery of the inhabitants, it was also thanks to Archimedes. He employed his knowledge fully in the service for the Syracuse army. He constructed cranes that could catch and turn over the Roman ships and the legend has it that he could also put those ships on fire. Here, various authors disagree whether it is a fact, or a myth. If we stick with the first version, then Archimedes probably used the fact that the shields of soldiers are in fact mirrors. Then all it took to put the ships on fire was to set those "mirrors" up in such a way that their foci met on a Roman ship, where the solar energy was thus also concerned.

In 212, the town of Syracuse lost and in connection with the invasion of the victorious army into the city, the life of Archimedes came to its end as well. Although Marcellus issued the command that the life of this genius should be saved, probably in the hope that he could serve Rome, that was not to be and the life of Archimedes ended when his town of birth lost its freedom. The legend says that exactly in the moment when the invaders entered the house of Archimedes, he was engaged in a problem concerning circles while he drew those circle in the sand. Roman soldiers took his request not to disturb his circles as a sign of resistance and one of them pierced Archimedes with his spear. Sources do not provide the name of this soldier and the name of his commander is only known to those well versed in the history of Rome. However, the name of Archimedes was not forgotten. Apart from his principle, we also know the screw of Archimedes and the spiral of Archimedes.

7.2 Lazare Carnot

In mathematics for elementary or high schools, we do not meet the name Carnot. We have nevertheless decided to include him in this selection. We will talk about the reasons for this later. Lazare Nicolas Marguerite Carnot was born on May 3, 1753 in a middle-class family in the town of Nolay in Burgundy. Since his youth, he demonstrated great talent for mathematics and technology, and thus it is not surprising that he graduated from the prestigious school in Mezieres, where *Gaspard Monge* also taught. After graduation in 1773, he became an officer with the French army, he served with various garrisons and he later became a captain the royal army.

When Bastille fell on July 14, 1789, he put all his abilities in the service of the revolution. Politically, he was close to the Jacobites, and besides, he had known Maxmilien Robespierre and this charismatic man had great influence on Carnot's opinions. Carnot's big moments came in 1793 when he became a member of the committee of public safety and was responsible for the army. During a short period of time, he performed a reform of the army and the French revolutionary forces defeated the intervention armies of Austria and Prussia. Among other things, he introduced universal conscription for young men from the age of 18 to 25, he eliminated noblemen, but also the illiterate, from the positions of officers and he also joined the voluntary units with regular army. We cannot forget also the founding of typography office from which military headquarters was later formed. He more than deserved the honorary title the *Organizer of the Victory*, with which he was decorated.

Carnot did not change his conviction, he especially protested against absolutist tendencies and non-democratic methods, and thus he had to flee into emigration several times. Although his opinions were close to those of Robespierre, his methods of government were alien to him and therefore he aided his defeat. For similar reasons, his relationship with Napoleon was rather precarious for the same reason, although they respected each other. Napoleon appreciated his abilities by promoting him to the position of division general and awarded him the title of an earl. Carnot then did not hesitate to support Napoleon upon his return from Elba and accepted the position of minister of the interior. The Bourbons did not forgive him his service during the revolution, so after their restoration, he had to leave France for good. He lived in Magdeburg for the rest of his life, where he worked as a scientist and also an advisor of the Prussian government for the issues of military schools and fortress construction. He died on August 2, 1823 and was buried in Berlin. In 1889, the French government decided that his remains be transported to France and stored in Pantheon.

Carnot is one of the most important mathematicians of his time. His field of interest was mainly geometry, e.g. Carnot's theorem. In his time, he was also an expert on fortress construction.

The name "Carnot" should be known to any high school student who studied physics, or rather thermodynamics. However, Lazare's share in this discovery is only indirect. Carnot's cycle and other discoveries are due to his son, *Nicolas Léonard Sadi Carnot (1796-1832)*, who inherited father's genes for science. In 1887, *Marie Francois Sadi Carnot (1838-1894)*, the great-grandnephew of the Organizer of Victory, was elected French president.

7.3 René Descartes

René Descartes was born on March 31, 1596 in la Haye in France. He was educated in a Jesuit college La Fleche in Anjou, where he was sent as an eight-year old. In his youth, he travelled a lot and also served in the army. He was in the army of Maxmilian of Bavaria, fighting against Bohemian nobility, but there is no evidence for his participation at the battle of the White Mountain. In 1628, he settled in Holland, where he lived for more than 20 years. In 1649, he could not resist the invitation of the outstanding woman on the Swedish throne, Queen Christina, and went to see her in Stockholm. The tough northern weather, however, was not good for his health and he died of pneumonia on February 11, 1650.

Descartes belonged to the greatest scientists of his time. In contrast to his scientific colleague and rival Fermat, he was not afraid of publishing. In 1637, he published his *Discours de la méthod pour bien conduire sa raison et chercher la vérité dans les science*. This book contained three supplements, from which the most important is the last one of those, entitled *La Géometrie*. It was in this supplement that he published the elements of analytical geometry, i.e. the part of mathematics that solves geometrical problems in an algebraic way. He also succeeded in simplifying mathematical symbolism, denoting the unknowns by the letters from the end of the alphabet and the constants by the letters from the beginning of the alphabet comes from him. We also have Cartesian coordinate system, but here, the primacy belongs to Pierre de Fermat. Descartes also solved problems in physics, especially in mechanics and optics, and he a.o. gave a full theory of the rainbow.

Descartes is naturally very well-known also to those interested in the humanities, because he is one of the best known philosophers and he also devoted time to psychology. Maybe it was success in mathematics that led Descartes to introduce similar methods also in philosophy; this direction is called rationalism. Its basis is his famous statement *I know, therefore I am.* He distinguishes between God as the uncreated and infinite substance and two created substances - the world of the body and the world of the mind with the attribute of thinking. He tried to formulate a general method of recognition, based on four principles: 1) accept only those things that are clear and apparent and without any doubts; 2) divide every problem into simpler parts that can be recognized easily; 3) proceed from the simple to the difficult; 4) put together complete lists and general overviews, so that it would become clear that we have not forgotten anything.

7.4 Leonhard Euler

Leonhard Euler was born on April 15, 1707 in Basel. His father was an evangelic priest and wanted his son to follow him. Luckily, this did not happen and the world did not lose one of its greatest mathematicians of all time. Johan Bernoulli, his teacher at the university in Basel, is to be credited with this. Although he was Swiss by birth, described in sports terminology, he played for Russia and Prussia. The emperors of these countries namely realized the importance of science and therefore they could draw the best scientists to the prestigious workplaces.

Apart from talent for mathematics, Euler also had phenomenal memory and imagination, and thus he could work until the end of his life, although he lost sight. His productivity was admirable: thanks to his manuscripts, the printers in St Petersburg had work even dozens of years after his death. However, he was far from the stereotype of a mathematician as an absent-minded man who lives only for his science and is not fit for normal life. It is said that he was a jovial man who lived a normal life.

As he touched upon all areas of mathematics with his work, it is very difficult to assess his work on such a small space. His work contributed to the development of number theory. In his work, he basically went in Fermat's footsteps and he succeeded in proving or disproving most of his statements; he for example proved the Last Theorem of Fermat for $n = 3$. He extended or generalized many of Fermat's findings, for example, he extended Fermat's little theorem for composite modules. His solution of the problem of seven bridges of Königsberg became a foundational stone of a new discipline - graph theory [Sa].

The core of his work, however, lies in mathematical analysis. His definition of the function z from 1755 extends the notion of a function from a mere analytical expression to a mutual dependence of two quantities. *When some quantities depend on others in such a way that when we change them (the second ones), they also change, we say that the first ones are functions of the second ones. This name has an outstandingly broad character and includes all possible ways of expressing one* *quantity through other quantities.*

7.5 Pierre de Fermat

Toulose judge Pierre de Fermat belongs to the most notable mathematicians in the world. He is usually called the prince of the amateurs, but in his times, mathematics as a profession did not exist, and therefore this denotation is slightly misleading. He was born in 1601, at least according to the publications until today. Fermat's father was a rich merchant with leather and he gave his son good education of a lawyer. Pierre chose his distant relative Luisa de Long as his lifetime partner. Together, they had two sons and a daughter. More than two centuries before the first aria of Kecal was performed, Fermat also counted what he could gain. Thanks to his good financial situation, he could buy a judge office in Toulouse, where he spent all his life. He divided his time between the office, family, and his great hobby, mathematics.

It is not known exactly when he started to devote time to mathematics, but it is certain that roughly in the 1630s, he joined the circle around Father Mersenne. In the time of Fermat, there were no scientific journals and no scientific conferences were organized. Scientists therefore had to publish their findings in book, which Fermat, for various reasons, rejected. There was a further option, correspondence with colleagues, and Mersenne improved this possibility through becoming a sort of a scientific secretary of a group of scientists and an organizer of a permanent corresponding scientific conference. For Fermat, this exchange was very convenient, because it allowed him to stay detached from the world.

A saying tells us that each man should bring up a son and for Fermat, fulfilling this obligation was a key one. This man, above all, paid attention to his job and science was only a hobby for him. Although he corresponded with the European scientific elite of that time, he did not take extra care of keeping his correspondence. His elder son Samuel succeeded his father in the office of the judge and he was also interested in mathematics, but did not inherit his father's talent. Nevertheless, he is to be credited with collecting the preserved letters from his father's inheritance as well as from his correspondents and he published these letters. If it was not for Samuel Fermat, the name of Fermat would not be known to historians of mathematics, or perhaps it would be included only in the most detailed books. Mathematics would also be deprived of one of its greatest mysteries, called in Czech *Fermat's great theorem* (in English known as *Fermat's last theorem*).

Fermat also had at his disposal Bachet's edition of the Arithmetics by Diophantus and this book interested him so much that he not only studied it in detail, but also wrote his notes and commentaries directly in this book. One problem solved by Diophantus concerned Pythagorean triples and Fermat wrote the following remark: *It is impossible to divide a cube into two cubes, biquadrate into two biquadrates, and in general any power into two powers. I found a beautiful proof for this fact, but this margin is too narrow.* In our contemporary mathematical language, the Diophantine equation $x^n + y^n = z^n$ does not have a solution in integers for $n > 2$. The simple formulation and Fermat's bold statement that he has a proof was an enormous challenge and since its publication until these days, both professionals and amateurs tried to find the lost proof. Wiles did indeed prove this hypothesis in 1994, but he used such methods that could never have been available to Fermat. Thus even after the proof has been published, attempts to find Fermat's beautiful proof did not cease and Fermat's last theorem has become the same thing for mathematics as the perpetuum mobile for the physicists. Today, most researchers are inclined to believe that Fermat incorrectly generalized his findings about certain concrete values and that this theorem cannot be proved in elementary mathematics.

Fermat is considered the founding father of number theory. Apart from Fermat's last theorem which has just been mentioned, we also know Fermat's little theorem, $a^{p-1} \equiv 1 \mod p$, where p is a prime and $(a,p) = 1$. He also paid attention to numbers of the form $F_n = 2^{2n} + 1$ (Fermat numbers), about which he mistakenly assumed that they are primes, but he did not have a proof for this. This problem has not been solved until today, but so far no primes were found, except for the first five that were known to Fermat already; on the contrary many of those numbers were factorized [Le].

Fermat belonged to the greatest figures of world science in the first half of the 17th century. He laid the foundations of analytical geometry and only because of the fact that Descartes published his ideas in a book, while Fermat's ideas were only known to a narrow circle of people, we now have Cartesian coordinate system, and not Fermat's. His correspondence with Blaise Pascal laid the foundational stone of probability theory. He knew the quadrature of the parabola, he could write the equation determining the tangent to a curve, and he is the founder of modern number theory. He could solve problems in mechanics and optics and we can find *Fermat's principle*, describing the spread of light, in high school textbooks for physics.

Fermat was a source of inspiration for mathematicians. It may seem that after his famous hypothesis has been solved, there is nothing that can surprise us about this Frenchman. However, towards the end of 20th century, a German historian of mathematics realized that the age given on Fermat's tomb does not comply with the dates given in literature and confirmed by registers. As we cannot assume that Samuel did not know his father's age, here is a possible explanation: Pierre born in 1601 died as a small boy, and at the same time, his father became a widower. He married for the second time and had a son with his second wife, whose name was again Pierre. The registers for this period are lost, so we cannot prove this hypothesis.

7.6 Johann Carl Friedrich Gauss

Gauss was born on April 30, 1777 in Braunschweig as the only son of a bricklayer and a water master. He was very gifted and if he attended school these days, he would probably be labelled as hyperactive by the psychologists. It is said that the teacher gave him the task of adding up numbers from 1 to 1000 in order to keep him busy for at least some short period of time. What a surprise it was for the teacher when the little Johann announced the correct result 500 500 after a short while. He namely noticed that the sum of the first and the last number is the same as the sum of the second and penultimate number and so on. It thus suffices to add the first and the last number and multiply this sum by the number of such pairs, or, if you like, to use the formula for calculating the sum of the first n members of an arithmetic formula $s_n = \frac{a_1+a_n}{2} \cdot n$. This story is well known, but it is less well known that as a three-year old, Gauss drew his father's attention to the mistake in the calculation of wages for the workers.

We can also meet the name Gauss in mechanics, electrostatics, magnetism, and astronomy, but we will focus on his contribution to mathematics. When complex numbers are taught, everybody will probably think of Gaussian complex plane, or the one-to-one mapping of complex numbers and points in the place. When we have experimental data and want to present them in a graphical way, we will use the method of least squares. This method allows us to construct such a curve from the measured values that the sum of the squares of the differences was the smallest possible. He also dealt with the construction of regular polygons and proved that Euclidean construction is only possible for $n = 2^k F_1 F_2 \cdots F_n$, where F_i are Fermat's primes (see the section on Fermat). Therefore it is possible to construct a regular 17-gon, whose construction Gauss found, but not a regular heptagon, although 7 is regarded to be a lucky number.

His *Disquisitiones artihmetiquae* belongs to key works in number theory. He also introduced the notion of congruence and proved the law of quadratic reciprocity. He also investigated the fifth postulate and concluded that apart from Euclidean geometry, other geometries may exist, but he did not publish his thoughts in this direction. His fruitful life came to an end on February 23, 1855 in Göttingen, where had taught at the university for many years.

Chapter 8

Czech mathematicians

8.1 Mathias Lerch

Mathias Lerch was born in Milínov by Sušice on February 1860 in a family of a minor farmer. His life was influence by an accident (fall from the loft), when he seriously hurt his leg, which remained bent in the knee, and thus the young Mathias could only walk with crutches. He only started attending school when he was 9 years old when his family moved to Sušice. Already at elementary school, the young Mathias demonstrated his talents, so it is no wonder that he decided to study first a secondary school and then the Czech Technical University in Prague. The consequences of his injury, however, haunted him all the time. Given his physical deficiency, he could not become a teacher at a grammar school, as he had wanted. He therefore decided for the career of a mathematician and a university teacher, which was very lucky for Czech mathematics. As a grammar school teacher, he probably would have been only mediocre, but in mathematics, he was very successful. After studying at Charles University and then for one year in Berlin, he became assistant at Czech Technical University in Prague. At that time, his publication activity started and he published in both Czech and foreign journals.

In 1895, another significant change happened in his life. As he could not go on being an assistant, he tried to obtain professorship at a university in Czech lands. His efforts, however, were not successful, which was strange, because Lerch was number one among the living Czech mathematician at that time already. It is possible that envy played a role therein and also the fact that Lerch adequately boasted about his qualities. Luckily, an offer from the university in the Swiss Fribourg appeared which Lerch accepted and was appointed professor at that university. The ten years that he had passed in Switzerland were probably the happiest years in Lerch's life, both in the professional and the personal one. In 1900, he went through a difficult operation which was nevertheless successful and Lerch could exchange the crutches for a walking stick. In that year, his niece Růžena Sejpková came to visit him (Lerch married her in 1921) and started taking care of his household matters. At that time, also Lerch's publishing activities culminated and his work was awarded the prestigious Great Prize of the Paris Academy of the Sciences in 1900. Lerch was the first Czech mathematician to receive this prize and a second Czech (after Purkyně).

Despite all his success, Lerch longed for return to his home country. He then stayed in Brno, as he became the professor of mathematics at the Czech technical university in Brno in 1906. When the university was founded in 1919, it was rather natural that he became the first professor of mathematics there and that he thus finally had a position that was adequate to his qualities. However, he did not enjoy this position for long, since he got a cold after swimming in the river and died of pneumonia in Sušice on August 3, 1922.

Lerch was unquestionably an outstanding mathematician, but his personality was rather controversial. Maybe it was a consequence of his childhood injury and his bad health condition in later years, because he suffered from diabetes, at that time an incurable illness. In his action, he was direct and would support his views even in spite of conflicts. Already in his youth, he had to go from Pilsen to Rakovník after his conflict with a catechist. His difficulties with finding a position have already been mentioned. On the other hand, he was thought of highly and was taken seriously.

Lerch was strongly influenced by Kronecker and his work was in the style of this German mathematician. He solved concrete problems, although he published more than two hundred papers, he never wrote a monograph, although his qualities in some areas of mathematics (elliptic integrals, malmsten series, quadratic forms) were outstanding and he would definitely be able to write the monograph. Although he was not a good teacher as far as the form is concerned (feeble and monotonous voice etc.), his lectures were excellent from the point of view of content. He allegedly said that when he had not been good for Czech schools, they were then not good for him and did not deserve his monographs. Despite that, Lerch has put his mark onto Czech schools and especially in the 1970, when set theory became the basis of teaching, he was the indirect nightmare of the parents of the schoolchildren, as it is almost certain that he is the author of the Czech word for set.

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