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| Polynomials |
| Text for Students of Mathematics Teaching |
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**Polynomials**

Text for Students of Mathematics Teaching

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Table of Contents

[2 Polynomials 5](#_Toc409954341)

[Chapter Objectives 5](#_Toc409954342)

[Time to Study 5](#_Toc409954343)

[Study Guide 5](#_Toc409954344)

[2.1 Algebraic Definition of Polynomial 6](#_Toc409954345)

[2.2 Polynomial Division 8](#_Toc409954346)

[2.3 Horner’s Method 10](#_Toc409954347)

[2.4 Application of Horner Scheme for Taylor Expansion 12](#_Toc409954348)

[2.5 Divisibility of Polynomials 18](#_Toc409954349)

[Exercises for Check 26](#_Toc409954350)

[3 Algebraic Equations 27](#_Toc409954351)

[Chapter objectives 27](#_Toc409954352)

[Time to Study 27](#_Toc409954353)

[Study Guide 27](#_Toc409954354)

[3.1 Roots of Polynomial 28](#_Toc409954355)

[3.2 Polynomial Derivative 31](#_Toc409954356)

[3.3 Polynomials with Integer Coefficients 33](#_Toc409954357)

[3.4 Vieta’s Formulas 36](#_Toc409954358)

[3.5 Linear Equation 39](#_Toc409954359)

[3.6 Quadratic Equation 39](#_Toc409954360)

[3.7 Cubic Equation 40](#_Toc409954361)

[3.8 Fourth Degree Equation 41](#_Toc409954362)

[3.9 Higher Degree Equations 41](#_Toc409954363)

[3.10 Selected Types of Algebraic Equations 42](#_Toc409954364)

[Exercises for Check 47](#_Toc409954365)

[4 Multivariate Polynomials 48](#_Toc409954366)

[Chapter Objectives 48](#_Toc409954367)

[Time to Study 48](#_Toc409954368)

[Study Guide 48](#_Toc409954369)

[4.1 Elementary Symmetric Polynomials 51](#_Toc409954370)

[Exercises for Check 57](#_Toc409954371)

[References 58](#_Toc409954372)

# Polynomials

## Chapter Objectives

Having studied this chapter you will be able to:

* define polynomial
* divide polynomials
* perform Taylor series expansion
* find the greatest common divisor of polynomials
* find roots of polynomials
* use Vieta’s formulas for searching for roots of polynomials

## Time to Study

Circa 3 hours

## Study Guide

The notion of polynomial must be already known to you from a high school. A polynomial (also known as a multinomial) was seen as a mathematical expression, which is expressed as a sum of several monomials. A monomial consists of several parts – a **coefficient** (which can be any real number) and one or more **variables** with a positive integer exponent.

In a mathematical analysis course you have encountered a polynomial as a special case of a (real) function (of a single real variable), id est as a mapping from the set of real numbers into the set of real numbers. These functions are very pleasant to work with due to their properties: their domain of definition is always the whole set of real numbers, they are continuous on this whole set and have derivations of any order which are polynomials again, etc.

At elementary school you have learned to add polynomials and to multiply them. At high school your skills have been extended by dividing polynomials.

From the algebraic point of view it is especially interesting that the result of adding and multiplying polynomials is again a polynomial. Thence polynomials form an algebraic structure with two (binary) operations.

Let us remind the form in which you are used to writing polynomials. Any polynomial can be transformed into the form

where .

In this text we will extend your knowledge in the field of polynomials by important concept definitions and by other possibilities of polynomials manipulation.

We will use concepts and knowledge you have already encountered in previous algebra courses. Proofs of theorems will be included only in case the proof is not overly demanding and there can be a certainty that the student will be able to follow the line of the proof. End of a proof will be marked by a square.

When constructing polynomials we have basically two ways we can proceed with. We can regard them as functions, which you have come across with in mathematical analysis. In this field a polynomial is perceived as a function of a single variable , which can be in the whole domain written as , where are constants. This definition states that every polynomial can be expressed as a finite sum of monomials in the form , where the variable is exponentiated to a nonnegative integer.

Yet we can also look at them from the point of view that a polynomial is defined by its coefficients and that the polynomials equality, addition and multiplication can be described by equalities and operations of their coefficients. We will use this second approach and will base the study on the algebraic definition of polynomial.

## Algebraic Definition of Polynomial

1. By a **polynomial** over a field T[[1]](#footnote-1) we mean a sequence ) of elements of the field T, in which there is at most a finite count of nonzero elements. The set of all polynomials over T is denoted T[x].

The polynomial is called a **zero polynomial**.

**Note:** For instance a polynomial can be identified with the sequence (2, 0, 0,-2 ,0 ,0 ,13 ,5).

1. Let us define over the set T[x] that for , :
2. ,
3. , , where , etc.

**Note:** E.g. the polynomials and are multiplied in this way:

hence , etc. The polynomial can be written in the form .

1. is an integral domain[[2]](#footnote-2) for each field .

*Proof:*

* Addition of polynomials is commutative, i.e. .
* Addition of polynomials is associative, i.e. .
* The zero element is the zero polynomial.
* The inverse element for a polynomial is a polynomial .
* Multiplication of polynomials is commutative.
* Multiplication of polynomials is associative.
* The identity element is the polynomial .
* Zero divisors: it is obvious that if polynomials *f* and *g* are nonzero then their product is nonzero too. 🞏

**Note:** Following is the demonstration how a polynomial represented using a sequence can be turned into the expression familiar from high school.

Let the polynomial identify with an element ,

**.**

Next let the polynomial be denoted as , as etc. Then there applies

.

1. Let . Then the **degree of a polynomial** () is defined in this way: if , , , then it is defined
2. For degrees of polynomials there applies:
	1. ,

*Proof:* The proof of the first part is obvious, thus part 2: , , , , and it is apparent that the product is nonzero. Hence the degree of the polynomial is . 🞏

**Note:** Let us consider the polynomials , , , a polynomial , , , .

## Polynomial Division

For polynomial division the knowledge of integer division will be applied. Surely, you can remember the algorithm to divide for instance a five-digit number by a two-digit one. The process of polynomial division is very similar.

Just as when dividing two integers over the integer domain the result can be either without or with remainder, the result of division of two polynomials can likewise be a polynomial without a remainder, or so-called partial quotient and a remainder.

**Note:** Let us remind the theorem regarding integer division with a remainder:

For each pair of integers , where , there exists exactly one pair of integers such that there applies:

1. ,
2. ⏐*b*⏐.

For instance, we can consider numbers . We know that , thus the partial quotient is 10 and the remainder 1. This fact can be expressed as

The form could also be chosen to be used, but according to the previous theorem, the remainder of division by five can attain values 0, 1, 2, 3 or 4.

The theorem regarding polynomial division is analogical. Before it is introduced let us remind the algorithm for dividing two polynomials, well known from high school.

**Note** (Algorithm for Dividing Two Polynomials): Before dividing two polynomials and , it is first necessary to check whether both of them are ordered by their power descending. We divide the first term of the polynomial by the first term of the polynomial , with the result of . Then we multiply the polynomial with the polynomial and subtract the resulting polynomial from the polynomial . The polynomial will be denoted as . If we continue with dividing. We then divide the first term of the polynomial by the first term of the polynomial again and proceed analogically. The succession of steps will be illustrated on the following example.

1. Divide polynomials , when , .

**Solution:**

Hence the result of division is a partial quotient and a remainder .

1. (Polynomial Long Division) For each pair of polynomials , where , there exists exactly one pair of polynomials for which there applies:
	1. ,
	2. .

*Proof:* Firstly, the existence of polynomials will be proved:

1. When , then .
2. When , then we follow the division algorithm.

. If , then

,

.

By adding these equations together we get , and by denoting and , we come to . By this we have proven the first part of the theorem.

Secondly, there will come the proof that there exists exactly one such pair of polynomials. The proof will be performed by contradiction, let us therefore assume that there could exist two different pairs of these polynomials:

, , .

By subtraction we get . It is then apparent that and since , there must apply , thus and from the validity of the equation there must also apply . We have proven that it is the same pair of polynomials, which is contradictory with the initial assumption. By this the proof has been performed. 🞏

1. Solve individually: Compute , for ,
	1. ,
	2. .

**Result:** a) , b) .

## Horner’s Method

Let us say that our task is to find the value of the polynomial for . We can certainly substitute 4 for , but let us assume we are indolent to exponentiate , and therefore we need some other approach with only simpler expressions involved.

The procedure will be as follows: Factoring out from the first two polynomial terms, we get . Then, from this newly formed expression and the third term of the polynomial, we factor out , resulting into . From this expression and the fourth term, when factoring out , we get . Altogether the original polynomial will gain the form

and by substituting with 4 we get .

The previous example can be generalised. When then there exists a polynomial and there applies

This situation can be expressed in a table called a **Horner scheme**:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | … |  |  |
|  |  |  | … |  |  |

When expanding the right side of the previous equation (you can perform by yourselves to ascertain the validity of the statement), we get:

We are interested in the number , which is the value of the given polynomial at the point . Therefore we transform the previous sequence into

1. Determine the value of a polynomial for .

**Solution:** The value of the polynomial at the given point will be found by using Horner’s method. We will proceed according to the fore-mentioned sequence

In our case, we place the polynomial coefficients into the first row of the table and the value into the first column. Then we write the first coefficient into the second row and calculate on: .

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | 1 | -3 | 2 | -2 | 16 |
| 5 | 1 | 2 | 12 | 58 | 274 |

That means we have found out that the value of the polynomial at the point 5 is 274. Yet the result of the table can be interpreted in another way, too. When dividing the polynomial by the polynomial , we get the partial quotient together with the remainder . This fact can be verified by dividing them. The result can be expressed as .

Hence Horner’s method can be used for simple division of a polynomial by a linear polynomial, as demonstrated in the following example.

1. Divide the polynomials from the example 2 b) using Horner’s method.

**Solution:**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | 1 | 0 | -1 | 0 | 3 | -5 |
| 2 | 1 | 2 | 3 | 6 | 15 | 25 |

The result is therefore .

1. Solve individually: Determine the value of the polynomial at the point . Write in the form .

**Result:**

## Application of Horner Scheme for Taylor Expansion of Polynomial

You certainly can recall Taylor series that could be used in mathematical analysis to replace some functions in the neighbourhood of a given point. Let us remind Taylor’s theorem, which states that a function , having in the neighbourhood of a point derivations up to an order of , can be in the neighbourhood of the point expressed as

where is the Taylor series and is the Taylor series remainder. This way the approximate value of the given function in the point can be found.

In case the function is a polynomial, there exists a much easier way of finding Taylor series, which is by the application of Horner’s method.

1. Let . Then if

where , then the expression on the right side is called a **Taylor series (Taylor expansion) centred at** .

1. For each there exists a Taylor series of a polynomial centred at .

*Proof:* When , then . When , then after dividing the polynomial by the expression we get

which means that is a constant polynomial. Now when we leave the polynomial as it is, then multiply the polynomial by the expression , the polynomial by the expression , etc., up to multiplying the polynomial by the expression and when we sum up all the resulting polynomials, we come to the Taylor series centred at , i.e.

1. Given the polynomial , apply the Horner’s method to find a Taylor series centred at .

**Solution:** We are to rearrange the polynomial in the following way: . The coefficients of the polynomial will be written into the first line, and the centre c into the first column. In the first step we proceed like we did in Horner scheme higher above. The last number in the second row is . In the second step we do not work with coefficients of the original polynomial anymore, but with the coefficients of the polynomial , instead, i.e. with the numbers from the second row. The last number of the row is left out, processing only up to . In this way we advance onwards. The table has the following form:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | … |  |  |
| c |  | … |  |  |  |
| c |  |  |  |  |  |
| c |  |  |  |  |  |
| c |  |  |  |  |  |
| c |  |  |  |  |  |
|  |  |  |  |  |  |

The numbers are the sought Taylor series coefficients.

1. For the polynomial find a Taylor series centred at .

**Solution:**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  | 0 | 0 | 0 | 0 |
| 1 |  | 1 | 1 | 1 | 1 | **1** |
| 1 |  | 2 | 3 | 4 | **5** |  |
| 1 |  | 3 | 6 | **10** |  |  |
| 1 |  | 4 | **10** |  |  |  |
| 1 |  | **5** |  |  |  |  |
| 1 | **1** |  |  |  |  |  |

Therefore the result is

**Note:** There is a noticeable similarity of Taylor series coefficients from the previous example with the Pascal’s triangle coefficients:

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  | 1 |  |  |  |  |  |  |
|  |  |  |  |  | 1 |  | 1 |  |  |  |  |  |
|  |  |  |  | 1 |  | 2 |  | 1 |  |  |  |  |
|  |  |  | 1 |  | 3 |  | 3 |  | 1 |  |  |  |
|  |  | 1 |  | 4 |  | 6 |  | 4 |  | 1 |  |  |
|  | **1** |  | **5** |  | **10** |  | **10** |  | **5** |  | **1** |  |
| 1 |  | 6 |  | 15 |  | 20 |  | 15 |  | 6 |  | 1 |

Examine for yourselves whether for polynomials or their coefficients of a Taylor series centred at are also equal to the Pascal’s triangle coefficients. Examine further what do Taylor series with different centres look like, for instance for , .

1. Solve individually: For the polynomial find out a Taylor series centred at .

**Result:** .

1. Let . If

then the right side is called a **Gaussian interpolation polynomial**.

1. Let . Then there exist numbers such, that there applies . The polynomial can therefore be expressed in the form of a Gaussian interpolation polynomial with chosen numbers .

*Proof:* When dividing the polynomial by the expression then by the expression , etc., we get:

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

When we multiply the resulting polynomials back again with the expressions in the right column and sum them up, we come to . 🞏

1. Express the polynomial in the form of Gaussian interpolation polynomial, given .

**Solution:** We will use similar procedure as when seeking a Taylor series by application of Horner’s method. Into the left column we will put values successively.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | 2 | -3 | -5 | 2 | 7 | -9 |
| 1 | 2 | -1 | -6 | -4 | 3 | **-6** |
| 2 | 2 | 3 | 0 | -4 | **-5** |  |
| -3 | 2 | -3 | 9 | **-31** |  |  |
| 2 | 2 | 1 | **11** |  |  |  |
| -1 | 2 | **-1** |  |  |  |  |
|  | **2** |  |  |  |  |  |

The polynomial is then of the form

.

**Note:** The Gaussian interpolation polynomial can be used to find a polynomial of order specified by values at  points, as will be demonstrated in the next example.

Let us merely remind that two points uniquely specify a straight line, three points specify a parabola, and so forth. For determining of a first-degree polynomial we therefore need values of two points, for a polynomial of the second degree we need values of three points, and generally for finding out of a polynomial of degree there are points needed.

1. Find a polynomial for which there applies .

**Solution:** Since the sought polynomial will be apparently of the third degree, we can express it in the form of the Gaussian interpolation polynomial, where we will choose to substitute with values of independent variable ,[[3]](#footnote-4) resulting to .

In this form of the polynomial we substitute with and as according to the assignment there applies , we can replace with 3, thus , and we find out that .

Similarly continuing for , , after rearrangement we get , and then .

Next we set to , which results into . Lastly , the result is .

The form of the polynomial is hence . After expansion we get

Now you can check by yourselves whether the values from the assignment comply with the resulting polynomial.

1. Solve individually: By an application of the Gaussian interpolation polynomial, determine such that there applies
	1. ,
	2. .

**Result:** a) , b)

1. Let . Considering , where ,

If for , then the right side is called a **Lagrange (interpolation) polynomial.**

**Note:** Lagrange polynomial can be, similarly as Gaussian interpolation polynomial, applied to find a polynomial passing through given points on a certain interval.

1. Use the Lagrange polynomial to find the polynomial that passes through the points .

**Solution:** The form of the polynomial will be , where

After substitution we get . You can check by yourselves that the polynomial conforms to the assignment requirements.

1. Solve individually: Use the Lagrange polynomial to solve Example 11.
2. Solve individually: By an application of the Lagrange polynomial find a polynomial that passes through the points .

**Result:**

## Divisibility of Polynomials

1. Let . We say that **a polynomial divides a polynomial**  if and only if there exists any such polynomial for which there applies . Then we write ⏐.
2. (Basic Properties of Divisibility of Polynomials)

Let , . Then there applies:

* 1. ⏐
	2. For all    and all there applies: If and at the same time , then also .
	3. Let , then there applies: If and also , then as well.
	4. and at the same time if and only if there exists such nonzero element , for which there applies .

*Proof:*

Re 1)

Re 2) , . Then , upon denoting , , we get .

Re 3) If then and similarly if , then . Therefore , where . Hence .

Re 4) If , then . For the degrees of these polynomials there applies , where , , , so then .

Re 5) If and at the same time then and also . From the previous part of the proof we know that and also , which means . Since , it must be , so is a nonzero constant . 🞏

1. We say that a polynomial is a **common divisor** of polynomials if and only if there apply at the same time both and .
2. Let . We call the polynomial a **greatest common divisor** of polynomials , if and only if there applies:
3. ,
4. For all polynomials there applies: if and also then .

**Note:** The definition above implies that a greatest common divisor is the common divisor of both polynomials which is divisible by all of the common divisors.

**Note**: Following there is presented the sequence of equations called Euclidean division of polynomials, which can be used to determine a greatest common divisor of two polynomials. Earlier you have encountered the Euclidean algorithm for finding the greatest common divisor of two integers, which is similar in principle and which will be demonstrated on an example.

1. Use the Euclidean algorithm to find out the greatest common divisor of numbers 450 and 294.

The greatest common divisor of two numbers is the last nonzero remainder in the Euclidean algorithm, which is .

**Euclidean Algorithm for Polynomials:**

Let . We start by dividing the polynomial by the polynomial and then we repeatedly divide the divisor from the previous step by the remainder, continuing until there is a result with a zero remainder.

1. A greatest common divisor can be found for each pair of polynomials . One of these greatest common divisors is the last non-zero Euclidean algorithm remainder for the polynomials and .

*Proof:* From the previous equation it is apparent from the previous equation that , then from the next-to-last equation , and so onwards to the second equation from which , and from the first equation then , hence is a common divisor of the polynomials .

Now it only remains to be proven that every common divisor of polynomials divides . If is a common divisor of both polynomials then it is evident that from the first equation, from the second one, and so forth to from the last one. 🞏

**Note:** In accordance with the previous theorem, no greatest common divisor there exists only when .

**Note:** If are two different common divisors of polynomials , then according to the property 2) in the Theorem 6 there applies and according to 5) there exists a non-zero number , for which it applies .

There also applies that when is a greatest common divisor of two polynomials, then is their greatest common divisor as well.

1. When is a greatest common divisor of polynomials , then the set is the **set of all greatest common divisors** of the polynomials . We say these common divisors are **associated** with each other.

Among the greatest common divisors of the polynomials there exists exactly one whose coefficient of the highest degree term is equal to 1. This polynomial is called a **normalized greatest common divisor** and denoted as .

1. Find the greatest common divisor of following polynomials over : , .

**Solution:** Euclidean algorithm will be applied. Since will not change when replacing the polynomials  with the polynomials associated with them, we will first multiply by 2. Then we will divide.

thus . Normalizing the polynomial into we proceed to the next step, this time dividing by .

therefore . After replacing the polynomial with the normalized polynomial again we can proceed to the division, getting

i.e. . Hence . The discovered greatest common divisor is really over the complex numbers field, thus satisfying the assignment requirement.

**Note:** In some cases it is also possible to find a greatest common divisor of two polynomials by their decomposition. Let us, for this purpose, revise some basic polynomial decompositions:

* ,
* ,
* ,
* ,

etc.

1. A polynomial is called **irreducible[[4]](#footnote-5)** over a field if and only if and there can be found no polynomials such that there applies .
2. A polynomial is called **reducible** over a field if and only if there exist polynomials for which there applies .

**Note:** The role irreducible polynomials play in the integral domain is analogical to the role of prime numbers in the ring .

1. For each non-constant polynomial there exists a number and irreducible polynomials over such that there applies

where , are normalized polynomials. This decomposition is unambiguous, except for the ordering of the polynomials .

The theorem could be also formulated in a different way: Every non-constant polynomial over a field can be decomposed into the product of a constant and normalized irreducible polynomials.

*Proof:* Let .

* The polynomial is irreducible over , then we normalize it into the form .
* Or the polynomial is reducible. Then there can be found its decomposition into polynomials irreducible over . Upon rearranging them into the normalized form, we get .

Now there still remains to prove the unambiguity of the decomposition. The proof will be performed by contradiction. So let us assume that there exist two distinct decompositions of the polynomial :

The polynomials are irreducible over . If the polynomial divides the polynomial , then there must also apply . Then either , or the polynomials are coprime. In the case they are coprime, then among polynomials there must be one that is equal to . Analogically, we can reason for the other polynomials . Thus we reach the contradiction with the assumption that the decompositions are distinct. 🞏

**Note:** When the same irreducible polynomials in the decomposition are multiplied by each other, then

This decomposition is called a **canonical decomposition**.

An analogy can be found in the ring of integers. From now on we will consider only non-negative integers, negative integer could be obtained by multiplication . Every non-negative integer greater than 1 can be decomposed into a product of prime factors, , i.e.

For instance, number can be expressed in the form of a canonical decomposition as .

1. Over the field of complex numbers, all polynomials of the 1st degree are irreducible. All polynomials of a degree greater than 1 are reducible.
2. Over the field of real numbers, there are irreducible just all polynomials of the 1st degree and those of the 2nd degree that have a negative discriminant.
3. For each there exists a polynomial such that is irreducible over the field .
4. Find decompositions of the following polynomials over .
	1. ,
	2. ,
	3. ,
	4. .

**Solution:**

1. The polynomial can be decomposed using the procedure well known from a high school, i.e.

We can clearly see by now that over both and the polynomial is irreducible and over its decomposition is .

1. , which is the decomposition in both and . Over we can continue with rearrangement .
2. , which is the decomposition in both and . In the expression in the last parentheses can be decomposed just like in the part a), thus getting .
3. . It is apparent that the polynomial is irreducible over . In it is possible to continue, . In further decomposition of the polynomial is possible but we will not perform it due to the form of the expression.
4. Determine a greatest common divisor of polynomials when ,
	1. using the Euclidean algorithm,
	2. by decomposition.

**Solution:**

1. .

Now we normalize the remainder into the polynomial and continue with the algorithm.

 remainder. 0.

The last non-zero remainder is , which is one of greatest common divisors, and .

1. In some cases it is possible to find out common divisors and also greatest common divisors by decomposition of the polynomials.

,

,

.

1. Solve individually: Apply the Euclidean algorithm to determine when ,

**Result:**

1. Polynomials are called **coprime** if and only if .
2. Let . Then there exist such that .

Furthermore, if , then polynomials can be found such that .

1. Use the Euclidean algorithm to determine polynomials such that , je-li , .

**Solution:** The first step will be to find .

We will write this result in the form

Continuing with division,

We will again rewrite it in the form

Since , the last non-zero remainder is the polynomial and hence .

To determine the sought form of the greatest common divisor , we will proceed backwards in the equation sequence

When expressing from the second equation and substituting the first equation for , we get

whence it is already apparent that , .

Overall

Verify the validity of the decomposition by yourselves by expanding the right side of the equation.

**Note:** When searching for a greatest common divisor of polynomials in the form using the Euclidean algorithm, it is not possible to normalize the polynomials or use associated polynomials for the purpose of simplicity.

1. Let Then there applies
2. If and at the same time , then .
3. If , and , then .

*Proof:* Re 1: We know from the previous theorem that there exist polynomials such that , then obviously

When multiplying this equation with the polynomial , we get

It is apparent that and according to the initial assumption , therefore from the previous equation there must also apply that .

Re 2: If , then there exists such that . Since by the initial assumption also , then after the substitution for from the previous step, there also applies . Furthermore, we assume that , so we can employ the first part of the theorem, resulting into

## Exercises for Check

1. Define polynomial using the algebraic definition. Demonstrate how from this form there can be obtained the form you are familiar with from high school.
2. State the theorem on polynomial division with a remainder.
3. Define the Taylor polynomial.
4. Describe the Euclidean algorithm for division of polynomials.
5. Define common divisor of two polynomials and greatest common divisor of two polynomials.
6. While calculating the value of the polynomial, a student pressed the following buttons on a calculator:

Determine the polynomial the student was evaluating and the value of .

1. Verify that there applies . Assemble a Horner scheme corresponding to this equation.
2. Divide the polynomial by the polynomial .
3. Find out a polynomial that passes through the points .
4. Find a greatest common divisor of the polynomials , by both Euclidean algorithm and decomposition.

**Results:**

1. ,

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | 1 | 0 | 1 | -3 | 7 |
| 3 | 1 | 3 | 10 | 27 | 88 |

1.

# Algebraic Equations

## Chapter Objectives

Having studied this chapter you will be able to:

* define algebraic equation
* find roots of a polynomial
* review the procedures for solving of linear and quadratic equation
* generically solve equations of 3rd degree
* solve binomial and reciprocal equations

## Time to Study

Circa 3 hours

## Study Guide

1. By an algebraic equation we mean an equation in the form

 *,*

where for *.*

**Note:** Solving an algebraic equation stands for searching for roots of the polynomial on the left side of the equation. In the following text there will be demonstrated procedures for solving of selected cases of algebraic equations as well as algebraic inequalities.

This text is focused solely on the equations whose coefficients are real numbers. There also exist equations with complex coefficients, for which the polynomial on their left side is over the field of complex numbers. However, since you will not usually encounter these polynomials in your common practice, this text is limited to polynomials over the field of real numbers.

Yet it is worth reminding that a polynomial with real coefficients can have both real and complex roots.

## Roots of Polynomial

1. A number is called a **root of a polynomial** , if and only if .
2. **(Bézout’s Theorem)**: Number is a root of a polynomial if and only if

.

*Proof:* For any and an arbitrary there applies .

1. is a root, then
2. is a root. 🞏
3. A number is called a **k-fold multiple root of a polynomial** if and only if there applies:
4. does not divide .
5. What is the root multiplicity of number 2 for the polynomial ?

**Solution:** The Horner scheme will be used:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | 1 | -5 | 7 | -2 | 4 | -8 |
| 2 | 1 | -3 | 1 | 0 | 4 | 0 |
| 2 | 1 | -1 | -1 | -2 | 0 |  |
| 2 | 1 | 1 | 1 | 0 |  |  |
| 2 | 1 | 3 | 7 |  |  |  |

Thus number 2 is a 3-fold multiple root of the polynomial and there applies

1. Find the value of in the polynomial such that the number is its at least a 2-fold multiple root.

**Solution:**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | 1 | 0 | 0 | -*a* | -*a* | 1 |
| -1 | 1 | -1 | 1 | -(1+*a*) | 1 | 0 |
| -1 | 1 | -2 | 3 | -(4+*a*) | 5+*a* |  |

There must apply that , hence . When actually dividing the polynomial by the polynomial , we then get

**Note:** When searching for roots of algebraic equations, it might be useful to apply the following theorem, which is valid for algebraic equations with real coefficients.

1. Let

be an algebraic equation. Let be its roots (both real and complex).

Let *.* Then there applies:

For each index there applies the inequality *.*

The number of positive real roots is equal to the number of sign changes in the sequence of non-zero coefficients or it is lesser by an even number.

**Implication:** When all coefficients of a polynomial are positive real numbers, this polynomial has no positive roots.

1. Determine intervals in which there lie real roots of the equation .

**Solution:** , thus . Hence all roots of the equation lie in the interval .

There are four sign changes in the sequence of coefficients 2, 3, -7, 6, -11, 5, thus in the interval there lie either 4 or 2 roots.

1. We call a field to be **algebraically closed** if and only if every polynomial with the degree of 1 or greater has at least one root over the field .
2. The field of rational numbers is not algebraically closed. For example the polynomial has rational coefficients, its degree is 2, but its roots do not belong into the set of rational numbers. Similarly, the field of real numbers is not algebraically closed. For instance, the polynomial has real coefficients, its degree is 2, but it does not have real roots because .
3. **(Fundamental Theorem of Algebra)**: Every polynomial with complex coefficients and with the degree of 1 or greater has at least one complex root.

The fundamental theorem of algebra can be also formulated as follows: The field of complex numbers is algebraically closed.

**Note:** All known proofs of the Fundamental theorem of algebra are based on methods of mathematical analysis. Performing the proof would require application of some theorems from mathematical analysis, whose implication is the fundamental theorem of algebra. Therefore no proof will be included.

1. Let . Then the polynomial has roots over , provided that a fold multiple root is counted as roots. Therefore for each polynomial there exist numbers such that

*Proof:* According to the Fundamental theorem of algebra, there exists a root such that . If , then there exists a root of a polynomial such that , so , and so on. Eventually , , thus the polynomial is a constant one, . 🞏

1. Let . Then the number is also a root of the polynomial and its multiplicity is the same as of the root .[[5]](#footnote-6)

*Proof:* 1) Let be a root of a polynomial . Then . There only remains to be proven that

Hence is a root of the polynomial as well.

2) Let be a fold multiple root of a polynomial . We will prove that is also a fold multiple root.

As and , the polynomial and thus also the polynomial has real coefficients.

where the polynomial cannot have roots . Therefore is a fold multiple root. 🞏

## Polynomial Derivative

1. Let . Then the polynomial is called a **derivative** of the polynomial .[[6]](#footnote-7)
2. For the derivative of polynomials there applies:
	1. for

**Note:** The proof of this theorem has been performed in detail in mathematical analysis.

1. Let . Then there applies: When the number is a -fold multiple root of the polynomial , then the number is a - fold multiple root of the polynomial . When is a simple root of the polynomial then is not a root of the polynomial .

*Proof:* Let be a -fold multiple root of a polynomial . Then there applies

.

For the derivative of the polynomial there applies

.

We will prove by contradiction that is not a -fold multiple root of the polynomial . If divided , then there would apply . From the previous assumption for the derivative there would have to apply

.

After cancellation we get

.

Choosing we get after substitution

.

Since is independent on then , which means that and

*.*

That would imply that the number is a -fold multiple root of the polynomial , which is a contradiction with the initial assumption that is only a -fold multiple root of the polynomial . 🞏

**Note:** The theorem is not valid in the opposite way: If a number is a root of the polynomial , then it does not necessarily have to be a root of the polynomial . For example. , , and number 3 is a 4-fold multiple root of the polynomial .

1. Given the polynomial , find coefficients such that 1 is at least a double root.

**Solution:** If 1 is a double root of the polynomial then it is a simple root of the polynomial . Therefore we will find the derivative of the polynomial and determine the values of the polynomials at the point 1:

Now all that remains is to solve the following system of equations:

Having solved this system of equations, we learn that , so the form of the polynomial is

1. Solve Example 22 using the derivative.

**Solution:**

, , , . The result is the polynomial .

1. Let . Then the following propositions are equivalent to each other:
	1. is a fold multiple root of the polynomial f.
	2. and at the same time does not divide .
	3. There exists a polynomial such that and at the same time .
	4. .
	5. is a fold multiple root of the polynomial .

**Note:** When searching for multiple roots of a polynomial , we can adopt the following procedure:

1. Find .
2. If , then does not have any multiple roots.
3. If , then each root of the polynomial is also a root of the polynomial , and moreover it is its multiple root: its multiplicity is greater by 1 than its multiplicity for the polynomial .
4. Demonstrate that the polynomial has only simple roots.

**Solution:** We will proceed according to the procedure in the previous note. Seeking the derivative of the polynomial , we find it to be . You can verify for yourselves by application of the Euclidean algorithm that . Thus the polynomial has truly only simple roots.

1. Let . Then there applies: The polynomial has the same roots as the polynomial and all of the roots of the polynomial are simple.
2. Let the polynomials for example be , . Then the polynomial and .

## Polynomials with Integer Coefficients

1. Let . When is a root of the polynomial , then for any there applies:
	1. ,
	2. ,
	3. .

*Proof:* is a root of the polynomial, i.e. . Hence for the polynomial we can write

When multiplying this equation by , we get

* 1. Having transposed the last term to the right side of the equation and factored out from all previous terms, we get:

thus , and as both and are coprime, there must apply .

* 1. Similarly as in the previous step we can transpose the term to the right side of the equation and factor out from all following terms, so getting:

reasoning similarly as in step 1), we can conclude that .

* 1. The polynomial can be expressed in the form of the Taylor polynomial centred at :

It is obvious that When we now perform substitution of with the root , we get

after multiplication of the equation by the expression we get

Reasoning the same way as in the steps 1) and 2), we can come to the conclusion that , and since , there applies , that is . 🞏

**Note:** The last theorem can be used to find all rational roots of polynomials with integer coefficients. In fact, the application of the part 3) of the last theorem consists in choosing certain values for , which are numbers and , thus

1. Find all rational roots of the polynomial .

**Solution:** The roots being searched for will have the form where , .

Divisors of number .

Positive divisors of number .

Next we will create a table where in the first row there will be all possible values of , corresponding values of will be in the second row, and in the third one. Since , we will highlight all divisors of number 4 in the second row, and because of , we will similarly highlight all divisors of number in the third row. Candidates for roots will then be those numbers which will have both second and third row values highlighted.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  |  |  |  |  |  |  |  |  |  |
|  | **2** | 3 | 3 | **4** | 5 | 7 | 0 | **1** | **-1** | **-2** | **-1** | -5 |
|  | 0 | **-1** | **1** | **2** | **1** | 5 | **-2** | **-3** | **-3** | -4 | -5 | -7 |

Therefore the candidates for roots are the numbers: . We will verify these roots using Horner’s method:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  | 2 | 1 | -12 | -20 | -19 | -6 |
|  | 2 | 0 | -12 | -14 | -12 | 0 |
|  | 2 | 6 | 6 | 4 | 0 |  |
|  | 2 | 2 | 2 | 0 |  |  |

We can clearly see in the Horner scheme that

The polynomial has no real roots and all sought rational roots are therefore numbers .

1. Solve individually: Find all rational roots of the polynomial .

**Result:** Candidates for roots: , , rational roots are .

**Note:** The previous procedure can be employed to solving certain algebraic inequalities as well, as shown in the following example.

1. Solve the inequality .

**Solution:** A number is a divisor of number , a number is a divisor of . When substituting number into the expression on the left side of the inequality, we find out that the value is . We will term the left side of the inequality as and the discovered fact as , which means that is a root. For this reason we will not include into a table a row for , because divisors of zero cannot be found. Further on .

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | 6 | -1 | -2 | -3 | -6 |
|  | **2** | 3 | **4** | 7 | 0 | **-1** | **-2** | -5 |

After checking the candidates using Horner’s method, we find that the numbers are roots of the polynomial, and the assigned inequality can be rewritten into the form

Now we can proceed on using the number line and zero points (perform by yourselves). The result are all numbers .

## Vieta’s Formulas

Certain problems can be solved by application of relationships between roots and coefficients of a polynomial that are presented in the following theorem.

1. Let be all roots of a polynomial . Then there apply so-called **Vieta’s formulas:**

*Proof:* When are roots of the polynomial , there applies . As we expand the right side and compare the coefficients for the same powers of , we get Vieta’s formulas. 🞏

1. Apply Vieta’s formulas to find a polynomial whose roots are .

**Solution:** There are five roots, thus we know that . The coefficient can be arbitrarily chosen, so we choose . Then

After rearrangement we get . The polynomial has the form

We can verify the validity by expanding:

1. Solve individually: Apply Vieta’s formulas to find a polynomial whose roots are .

**Result:**

**Note:** Vieta’s formulas can be utilised for simple in-memory solving of some quadratic equations with integer roots. For an equation there applies , . For example, in case of the polynomial , for its roots there applies and . Upon solving this system of equations, we get .

1. Find out the coefficients and the form of a quadratic polynomial knowing that the product of its roots is equal to and its sum to .

**Solution:** According to Vieta’s formulas there applies:

Hence we know that and . The form of the fractions implies that , then and . Thus the form of the sought polynomial is .

1. There is given the quadratic equation . Let us mark its two distinct real roots with the symbols . Compute the value of the expression .

**Solution:** We will apply the knowledge of Vieta’s formulas

* 1. ,
	2. ,

thus , . When rearranging the expression , we get , and after substitution .

It was possible to solve the example also without Vieta’s formulas, computing the roots of the quadratic equation using the standard formula, but considering that the roots are not rational, the process of computation could prove to be rather difficult indeed, as you can check by yourselves.

1. Find all roots of the polynomial given the knowledge that for the roots of the polynomial there applies the relation .

**Solution:** According to Vieta’s formulas there applies:

When substituting the relation into the first equation, we get

From the second equation we get by rearrangement

Combining this rearranged equation with the fourth equation:

2)

4)

Now we perform the substitution

Returning to the original variables, we get .

Now we will solve the system of two equations and , perform the substitution , and having solved the equation

we come to two roots .

When we use an analogical process for the equations and , we get another two roots .

As we are dealing with a polynomial of the fourth degree, which has four roots, we have found all of its roots.

1. Solve individually: Find all roots of the polynomial , when among the roots of the polynomial there applies the relation .

**Result:**

Following are briefly described basic methods of solving some selected types of algebraic equations.

## Linear Equation

The term stands for an equation in the form . This equation has always a solution in the form

## Quadratic Equation

It is an equation in the form . You have already encountered this type of equation at elementary school where you were solving quadratic equations in a special form. You learned to generically solve quadratic equations at high school. The following text reminds how quadratic equations are being solved.

For the calculation of roots a quadratic equation is rearranged using the method of the complement to a full square.

Now if we mark the expression with the letter , from the formula for difference of squares we get

Discussing the last term: if the expression is to be zero, then either one or another of the expressions in parentheses must be zero, thus

As , , for roots of quadratic equation we get the well-known formula

The expression is called a **discriminant** and according to it being positive or negative the decision can be promptly made whether the equation has real or complex roots.

Let us just remind that in some simple cases with integer coefficients Vieta’s formulas are more suitable to be used to determine roots faster. For instance, for the equation there applies , so the roots are .

## Cubic Equation

The cubic equation is an equation in the form . The first step in its solution procedure consists of dividing the equation by the number and after re-labelling of the coefficients we get the following equation

We proceed by using substitution *.* After rearrangement, we get the equation in the so-called reduced form

Let us mark . Furthermore, let denote one (fixedly chosen) of the two values of the second square root. Next, let denote any (fixed) of the three cube roots , and lastly, let *v*denote one of cube roots for which there applies the relationship *.* Then roots of the equation are

The previous procedure is called **Cardan’s method**.

A cubic equation can have both real and complex roots. Their nature can be determined from the value of a **discriminant** which has the form . When being limited to a cubic equation with real non-zero coefficients (otherwise, when or , the solution is trivial).

1. : .
2. :one real root and two imaginary complex conjugate roots, specified by the Cardan formula.
3. :three real roots, however in this case it is not possible to find them using Cardan’s method (so-called “cassus irreducibilis”). The goniometrical solution procedure needs to be applied:

Firstly, we compute the value of the angle (one fixedly chosen) from the equation . The sought roots are then specified by the following relationships:

It is obvious that both Cardan’s method and the lastly-mentioned relationships are very laborious and cumbersome for practical calculations. Moreover, it is rather difficult to rearrange the acquired results into a usable form. For our purposes it is therefore more suitable to find the solution of a cubic equation in the way described in Chapter 3.3 which allows determining roots of an equation with integer coefficients.

## Fourth Degree Equation

It is an equation in the form . This equation is of the last degree that is generally solvable. Nevertheless, the general solution procedure for a 4th degree equation is toilsome and tedious to that extent that it is hardly ever used in practice, and thus it will not be presented in this text. For equations in the school practice some simpler ways of solving are sufficient (transformation of an biquadratic equation in the form by the substitution to a quadratic equation, procedures for solving binomial and reciprocal equations).

## Higher Degree Equations

For equations of the 5th and higher degree there does not exist any generic solving algorithm anymore [[7]](#footnote-8). According to the theory created by French mathematician Galois there exists for every *n* *≥ 5* an algebraic equation of the degree *n* that is not solvable by algebraic methods. Even though many equations of higher degrees can be solved, they must be in some special form. Either it is possible to “guess” the roots one at a time, or the equation is in the form that allows it to be solved as a binomial or reciprocal equation. Both these methods are being covered at high school and will be reminded in the following chapters.

Another method how to obtain roots of an algebraic equation, is to use numerical methods. Advances in information and computation technology contribute to the widespread use of these methods. The general procedure consists of the following three steps: assessing bounds of polynomial roots (according to Theorem 16), their separation, and approximation. Searching for roots of polynomials with real coefficients by applying these numerical methods relies on the observation that a polynomial is a continuous function in the domain of real numbers. Among contemporarily used methods there belong the Bisection method, the Fixed-point iteration, Newton’s method and Halley’s method. These problems will not be dealt with in the following text.

## Selected Types of Algebraic Equations

**Binomial Equation**

A binomial equation stands for an equation of the type . According to the Fundamental theorem of algebra there exist solutions of this equation. These solutions are in the form

and they form vertices of a regular planar gon. This gon lies in the Gauss plane on a circle that has its centre in the origin of coordinate axes and its radius . When searching for these roots, we proceed as follows:

1. Write down the sought root in the form .
2. Find the radius .
3. Find the angle for which there applies , thus , where . The angle can be then found from the position of the number in the Gauss plane. It is the angle that the position vector of the number contains with the real axis.
4. Solve the equation in . Depict its roots in the Gauss plane of coordinates.

**Solution:** First we rewrite the equation in the form .

* 1. We will write the sought roots in the form .
	2. .
	3. or , since number 2 lies in the Gauss plane on the positive part of the real axis. Then .

The solution are three roots in the following form:

We can see that the first root is a real one while the other two are complex. They form vertices of a regular triangle in the Gauss plane.



1. Solve the equation .

**Solution:** When finding a radius for the number , there applies . Therefore, in our example there applies .

The number lies in the fourth quadrant, in which case the angle can be determined as , where can be computed from the following right triangle:



. There are five roots for which there applies

1. Solve the following binomial equations individually:

**Results:**

**Reciprocal Equation**

A reciprocal equation is an equation of the form

where for , or for .

It is called reciprocal on the account of the observation that when is one of its roots then (the reciprocal value) is also its root.

The reciprocal equation of the **even degree** for an even can be distinguished from the one of the **odd degree** for an odd , just as the equation of the 1st class when , from the equation of 2nd class when .

The procedure for solving a reciprocal equation is as follows:

* Each reciprocal equation of the 2nd class has the root . Upon dividing it by the binomial , we get a reciprocal equation of the 1st class.
* Each reciprocal equation of the 1st class of an odd degree has the root . When divided by the binomial , the result is a 1st class reciprocal equation of an even degree.
* A 1st class reciprocal equation of an even degree can be transformed into an algebraic equation of the half degree by dividing it by the expression , and performing the substitution
1. Find roots of the equation .

**Solution:** It is a 2nd class reciprocal equation of an odd degree. We will proceed as follows:

* 1. We employ the Horner’s method to verify the roots . Number will be certainly one of the roots of the equation.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 2 | 5 | 0 | -3 | 3 | 0 | -5 | -2 |
| 1 | 2 | 7 | 7 | 4 | 7 | 7 | 2 | 0 |
| -1 | 2 | 5 | 2 | 2 | 5 | 2 | 0 |  |
| -1 | 2 | 3 | -1 | 3 | 2 | 0 |  |  |

We can see that is a simple root and is a double root of the equation. If these numbers were to be tested further on, we would not get zero in the last column of the Horner schema anymore, as you can check by yourselves. Hence we can rewrite the equation into the form

and we can notice the expression in the third parentheses form a reciprocal equation again, this time of the 1st class and of an even degree.

* 1. We rearrange the equation by dividing it by , writing the terms with the same coefficients together:
	2. Before we perform the substitution , there needs to be determined how to express :

When substituting the obtained expressions into the equation , we get

Let us return to the original variable. When , then , we multiply this equation with (assuming that and get the equation , whose solutions are the numbers .

When , then and with a rearrangement similar to the previous step we get the equation , whose solutions are .

The complete solution is represented by the set

1. Solve individually the reciprocal equation .

**Result:**

# Exercises for Check

1. Define algebraic equation and its roots.
2. State and prove Bézout's theorem.
3. Is the field of real numbers algebraically closed? Explain.
4. Write down Vieta’s formulas for a polynomial of the 5th degree.
5. Find out multiplicity of the roots for the polynomial .
6. Apply a derivative to find out multiple roots of the polynomial .
7. Given the quadratic equation , where are its roots, determine the value of the expression .
8. On the example of the polynomial explain how rational roots can be found for a polynomial with integer coefficients. Find all rational roots.
9. Solve the inequality on the set of real numbers.
10. On the set of complex numbers solve the equality .

**Results:**

1. is a triple root, is not a root.
2. is a triple root, is a double root.
3. .
4. No solution.
5. .

# Multivariate Polynomials

## Chapter Objectives

Having studied this chapter you will be able to:

* define multivariate polynomial
* determine the degree of a multivariate polynomial
* lexicographically order a multivariate polynomial
* define symmetric and homogeneous polynomial
* define elementary symmetric polynomials.

## Time to Study

Circa 2 hours

## Study Guide

1. A polynomial of variables stands for a sum of a finite number of terms

where , is the coefficient of the term , and are non-negative integers. A polynomial of variables over a field is labelled with the symbol . The **degree of a term**  stands for the number .

1. The degree of the term is 8.

**Note:** In the following text the notion of term height will also come handy. Let be a polynomial of variables and be its arbitrary term. Then the sorted -tuple of non-negative integers is called a **height of the term.**

1. **The degree of a non-zero polynomial** is equal to the highest of the degrees of its terms with non-zero coefficients. A polynomial whose all terms have the same degree is called a **homogenous polynomial** (of the degree ).
2. The polynomial is a homogenous polynomial of the degree 1, the polynomial is a homogenous polynomial of the degree 2, while the polynomial is a homogenous polynomial of the degree 3.
3. Terms , are called **similar terms** if and only if .

**Note:** When possible we sum up all similar terms outright. There applies .

1. Considering operations +, , the set is an integral domain (operations are defined generally, is a group and is a commutative semigroup).
2. Every polynomial can be written in the form of a sum of homogenous polynomials, each of them of a different degree. This expression is unambiguous (except for their order).

**Note:** In case of univariate polynomials the terms can be ordered in the ascending or descending manner with respect to the variable exponent. Therefore some other procedure must be defined.

1. Let , be two terms of variables. Let us state that the term comes before the term if there exists an index *,* for which there applies

The condition that the term comes before the term , or ,is denoted as *.*

1. The relation is a relation of a linear ordering over the set of all terms containing variables (the relation is irreflexive, antisymmetric and transitive).
2. The relation is called a **relation of lexicographical order** of terms of variables. When the terms of a polynomial are ordered by this relation, we can state that the terms of the polynomial *f* are lexicographically ordered. The term that comes before all other terms of the polynomial is called a **leading term of the polynomial** .
3. Determine a lexicographical order of the polynomial and its leading term.

**Solution:** Upon factorization we get , which is the lexicographical order of the polynomial. We can verify it using the term heights: The leading term of the polynomial is .

1. Let be two arbitrary real non-zero polynomials of *n* variables. Then the product of the leading terms of the polynomials and is the leading term of the product *.*
2. A polynomial is called to be **symmetric**, when it does not change after any permutation of variables, i.e. for any arbitrary permutation of indices there applies:

The set of all symmetric polynomials of variables over the field of real numbers is denoted as *.*

**Note:** A symmetric polynomial of two variables fulfils the condition , whereas a symmetric polynomial of three variables fulfils the condition . The polynomial is a symmetric polynomial of the degree 1, the polynomial is a symmetric polynomial of the degree 2.

1. Let be the leading term of a symmetric polynomial *.* Then there applies .
2. Find out a lexicographical order of the symmetric polynomial and its leading term.

**Solution:** We proceed by factorization, getting , which is the lexicographic order of the polynomial. The leading term of the polynomial is .

1. Let be a term consisting of variables. Then there exists only a finite number of leading terms of symmetrical polynomials of variables that come after the term *.*
2. Given the polynomial , determine all its leading terms.

**Solution:** The heights of the terms of the polynomial are respectively. When we order these heights by the relation we get . The leading term of is thus the term . However, the inequalities from Theorem 30 apply also for heights and , and hence there are other leading terms of the polynomial and .

We could conclude that the given polynomial is the sum of three elementary symmetric polynomials, whose leading terms are , , . The procedure for the decomposition of symmetric polynomials into a sum of elementary symmetric polynomials will be presented in the following chapter.

## Elementary Symmetric Polynomials

1. Polynomials from in the form:

are called **elementary symmetric polynomials** of variables.

**Note:** The elementary polynomial is therefore the sum of all variables *,* the polynomial is the sum of products of all pairs, the polynomial is the sum of triples, and so forth. Finally, the polynomial is the product of all variables *.* Considering the case and the familiar variables notation then:

Symmetric polynomials appear in Vieta’s formulas and in addition they can be used to solve some systems of equations which it is not possible to solve by methods known from elementary and high schools, as will be demonstrated.

The following is the **fundamental theorem of symmetric polynomials** which deals with the existence and unambiguity of an arbitrary symmetric polynomial expression, using elementary symmetric polynomials.

1. Every symmetric polynomial can be expressed as a polynomial of variables over , i.e.

and this expression is unambiguous.

1. Concerning power sums upon application of the binomial theorem

we get

,

,

,

and so on. The following algorithm can be generically applied to the transformation of a symmetric polynomial into the sum of elementary symmetric polynomials.

**Algorithm for Determining All Leading Terms of Symmetric Polynomials of the Degree , That Come After The Leading Term**  ()**:**

1. Write down the height of the leading term of the polynomial, i.e. the sequence of exponents .
2. Find in this sequence the last number greater than 1. Decrement this number by 1, leave the fore-going numbers as they are and select the numbers that go “after” as small as possible for it to apply and .
3. Repeat the procedure until obtaining the sequence 1, 1, …, 1.

Then the obtained sequence of the term heights describes all leading terms of all symmetric polynomials of the given order that come after the given leading term.

When seeking the symmetric polynomial in the form , to the first leading term we will create the corresponding term . The similar process is then used for other leading terms.

The polynomial can be written in the form , where is the coefficient of the first leading term and the constants are **undetermined coefficients** that can be found by substitution of concrete values for .

1. Transform the polynomial into the form .

**Solution:** Firstly, we set up a sequence of exponents and create corresponding leading terms of a general symmetric polynomial whose leading term is . Such polynomial should be of the form: . For each leading term we then create a corresponding term :

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| 2 | 2 |  |  |  |  |
| 2 | 1 | 1 |  |  |  |
| 1 | 1 | 1 | 1 |  |  |

Therefore, for the polynomial there applies

where and the coefficients will be computed by two choices of substitution for .

**Case 1:** , , , , , . After substitution into the equation we get

**Case 2:** , you can perform the substitution by yourselves to check that . Hence there applies

This leads us to the sought form of the polynomial

1. Transform the symmetric polynomial into the variables .

**Solution:** Upon factorization we would get Thus we can write down the sequence of exponents of leading terms, striking out the rows where the sequence has more than 3 terms, since the polynomial has only 3 variables.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 4 | 2 |  |  |  |  |  |  |
| 4 | 1 | 1 |  |  |  |  |  |
| 3 | 3 |  |  |  |  |  |  |
| 3 | 2 | 1 |  |  |  |  |  |
| 3 | 1 | 1 | 1 |  |  |  | **×** |
| 2 | 2 | 2 |  |  |  |  |  |
| 2 | 2 | 1 | 1 |  |  |  | **×** |
| 2 | 1 | 1 | 1 | 1 |  |  | **×** |
| 1 | 1 | 1 | 1 | 1 | 1 |  | **×** |

We have 4 undetermined coefficients, therefore 4 different choices will have to be performed (notice that for instance the cases (1,1,0) a (0,1,1) are equivalent with respect to the result, which is caused by the symmetrisation of the polynomial):

**Case 1:** ,

**Case 2:** ,

**Case 3:** ,

**Case 4:** ,

Now we will solve the following system of equations for the remaining undetermined coefficients:

The solution of this system is . To summarise

1. Solve individually: Transform the polynomial into the variables .

**Result:**

1. Solve individually: Transform the polynomial into the variables .

**Result:** The leading term ,

Symmetric polynomials have their use in application tasks. However, only two simple examples will be shown here to illustrate the usefulness of symmetric polynomials in various types of tasks.

1. Solve the following system of equations in the domain of real numbers

.

**Solution:** We can compute and check that and . Thus we get a new system of equations

After substitution of the second equation into the first one, we get the equation , by estimating and substituting into the Horner scheme stepwise we eventually find one real root . As , we will now be solving the system

whose solution is the set of two ordered pairs .

We can perform the check by substituting into the original assignment and having done this in both cases, we come to the identity .

1. A cuboid has edges of lengths . There applies and . Determine the volume of the cube that has the body diagonal of the same length as the given cuboid.

**Solution:** Let us mark the cube edge length with . The body diagonal of the cuboid is of the length while the length of the cube body diagonal is . This leads to the equation

which after exponentiation turns into

The polynomial is a symmetric polynomial and there applies , hence

At the same time we know from the assignment that and . Thus

After rearrangement , . The sought volume of the cube is then

1. Solve the following system of equations individually:

**Solution:** .

## Exercises for Check

1. Define polynomial of variables and its degrees.
2. Define homogeneous and symmetric polynomial.
3. Determine the degrees of the following polynomials. Are they homogeneous or symmetric?
	1. ,
	2. ,
	3. ,
	4. ,
	5. ,
	6. .
4. Apply the binomial theorem to rewrite the sum of powers by elementary symmetric polynomials.
5. Express the polynomial using elementary symmetric polynomials.

**Results:**

1. a. 1, homogeneous, b. 2, homogeneous, c. 3, symmetric, d. 2, neither homogeneous nor symmetric,, e. 3, homogeneous, f. 3, homogeneous and symmetric
2. .

# References

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1. By a numerical field we mean an ordered triple , where *T* is a subset of the set of complex numbers C, such that  and there applies:

(it is closed for both addition and multiplication),

 (it is closed for inverse elements),

(it is closed for inverse values of nonzero elements). [↑](#footnote-ref-1)
2. Let us remind that an integral domain is a commutative ring that contains the identity element and where the zero element does not have any non-trivial divisors. [↑](#footnote-ref-2)
3. Any three values of an independent variable can be arbitrarily chosen in any order, but this decision must be respected in the rest of the process and suitable function values must be substituted for $f$ respectively. [↑](#footnote-ref-4)
4. Irreducible stands for one that cannot be decomposed into a product of non-constant polynomials. [↑](#footnote-ref-5)
5. Let us remind what a number $\overbar{c}$, called a complex conjugate of the number $c$, stands for. When $c=a+bi, a,b\in R$, then $\overbar{c}=a-bi$. Furthermore, for numbers $c\_{1},c\_{2}\in C$ there applies: $\overbar{c\_{1}\pm c\_{2}}=\overbar{c\_{1}}\pm \overbar{c\_{2}}$, $\overbar{c\_{1}.c\_{2}}=\overbar{c\_{1}}.\overbar{c\_{2}}$, for $k\in N$: $\overbar{z\_{1}^{k}}=(\overbar{z\_{1}})^{k}$, for $a\in R: \overbar{a}=a$ [↑](#footnote-ref-6)
6. You have already encountered the derivative during your study of mathematical analysis. The derivative was defined generically for a function and could be applied to analyze various functions’ properties, such as local extremes of function, convexity and concavity, examination of its behavior and so on. [↑](#footnote-ref-7)
7. No universal formula can be found that would contain only coefficients, a finite number of arithmetical operations and a finite number of nth root operations. [↑](#footnote-ref-8)